Nonparametric Identification and Estimation of Double Auctions with Bargaining *

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Abstract

This paper studies the nonparametric identification and estimation of double auctions with one buyer and one seller. This model assumes that both bidders submit their own sealed bids, and the transaction price is determined by a weighted average between the submitted bids when the buyer's offer is higher than the seller's ask. It captures the bargaining process between two parties. Working within this double auction model, we first establish the nonparametric identification of both the buyer's and the seller's private value distributions in two bid data scenarios; from the case of all bids being available, to the case of only transacted bids being available. Specifically, both private value distributions are point identified when all of the bids are observed. They are, however, partially identified when only the transacted bids are available. A sharp characterization of the identified set is provided in the latter case. Second, we estimate double auctions with bargaining using a two-step procedure that incorporates both boundary and interior bias correction. We then show that our value density estimators achieve the optimal uniform convergence rate of first-price auctions. Monte Carlo experiments show that, in finite samples, our estimation procedure works well on the whole support and significantly reduces the large bias of an estimator without bias correction in both boundary and interior regions.

KEYWORDS: Double auctions, bargaining, nonparametric identification, kernel estimation, boundary correction.

JEL CLASSIFICATION: C14, C57, C78, D44, D82

1 Introduction

Bilateral bargaining is one of the most important forms of trade. Despite there has been a large literature of bargaining in theory and in laboratory experiments over the past sixty years, we have

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seen a burst of empirical investigations of bargaining only in the past decade. The latter is mainly due to the increasing availability of bargaining data to the academic community, see, e.g., the online bargaining interaction data from the eBay's best offer platform (Backus, Blake, Larsen, and Tadelis, 2020, Backus, Blake, Pettus, and Tadelis, 2020), the data of wholesale used-auto auctions (Larsen and Zhang, 2018, Larsen, 2021), and the data of union-management negotiations (Treble, 1987, 1990).

A bargaining framework with two-sided incomplete information allows for inefficient outcome which occurs in real-world trade but is excluded by a framework with complete information. As one influential candidate of the former, the double auction with bargaining (or k double auction) considers linear strategies for both buyer and seller. The linearity of strategies has been confirmed by experimental studies (see, e.g., Radner and Schotter, 1989, Schotter, 1990). This paper nonparametrically identifies and estimates the double auction model with bargaining. Our framework can be used to recover the buyer's and seller's updated value distributions based on the last round of bids, since the prior rounds of bids are usually used to reveal limited information about own reservation values (Parco, Rapoport, Seale, Stein, and Zwick, 2004). For instance, our method can be employed to estimate the buyer's and the seller's (most updated) value distributions by using the last round of offers from the bargaining data of eBay's best offer platform (or wholesale used-auto auctions, or union-management negotiations).

This paper contributes to the literature of non-cooperative bargaining games with incomplete information. On the theoretical side, such games have been extensively studied by, e.g., Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), Rubinstein (1985), Grossman and Perry (1986), Leininger, Linhart, and Radner (1989), Satterthwaite and Williams (1989, 1993), Brams and Kilgour (1996), Ausubel, Cramton, and Deneckere (2002), Kadan (2007), and Loertscher and Marx (2019), among others. In addition, there is also a large experimental literature which examines the theoretical properties of bargaining with incomplete information; see, e.g., Radner and Schotter (1989), Rapoport and Fuller (1995), Daniel, Seale, and Rapoport (1998), Rapoport, Daniel, and Seale (1998), Seale, Daniel, and Rapoport (2001), Parco (2002), and Parco and Rapoport (2004), among others. Empirically, there is a fast growing literature to investigate the role of asymmetric information in bargaining.[1] Examples with reduced-form approach include Merlo and Ortalo-Magne (2004), Scott Morton, Silva-Risso, and Zettelmeyer (2011), Backus, Blake, and Tadelis (2019), Backus, Blake, Larsen, and Tadelis (2020), Backus, Blake, Pettus, and Tadelis (2020), Bagwell, Staiger, and Yurukoglu (2020), and Grennan and Swanson (2020). Another line of empirical research conducts structural analysis of incomplete information bargaining in, e.g., wholesale used-auto market (Genesove, 1991, Larsen and Zhang, 2018, Larsen, 2021), and the market for local autorickshaw transportation (Keniston, 2011). Our paper belongs to the second research line (of structural approach) and provides an empirical methodology to use the data on offers and asks at the last round of the bargaining process to estimate the updated valuation distributions of both participating parties. Our method can be applied to the experimental data to test the k double auction theory in laboratory environment in the same flavor as Bajari and Hortacsu (2005). It can also be applied to the field data to quantify both the ex ante and ex post inefficiency introduced by private information (see Satterthwaite and Williams, 1989).

Our paper is also related to the literature which examines nonparametric identification and

[1] There is also a growing literature on structural analysis of bargaining with complete information. Examples include Merlo (1997), Diermeier, Eraslan, and Merlo (2003), Eraslan (2008), Merlo and Tang (2012, 2019), and Simcoe (2012).

estimation of one-sided auctions. This work was pioneered by Guerre, Perrigne, and Vuong (2000) for the identification and estimation of first-price auctions, and has been followed by many other papers. For comprehensive surveys, see Athey and Haile (2007), Hendricks and Porter (2007), Hickman, Hubbard, and Sağlam (2012), Gentry, Hubbard, Nekipelov, and Paarsch (2018), Perrigne and Vuong (2019), and Hortaçsu and Perrigne (2021). In identification part, we generalize the Guerre, Perrigne, and Vuong (2000)'s nonparametric identification strategy to the double auction setup. The model primitives are shown to be partially identified when only transacted bids are available, but to be point identified when the failed bids are also available. Our identification results are hence similar to Gentry and Li (2014), who obtained constructive bounds on model fundamentals which collapse to point identification when available entry variation is continuous in auctions with selective entry. There are other papers obtaining partial identification in the context of one-side auctions, see, e.g., Haile and Tamer (2003), McAdams (2008), Tang (2011), Aradillas-López, Gandhi, and Quint (2013), Komarova (2013), and Chen, Gentry, Li, and Lu (2020). Compared to this research line, however, we consider identification in a different auction setting (namely, double auctions with bargaining) which introduces not only asymmetric information but also asymmetric bidding strategies.[2] In estimation part, our paper is closely related to Hickman and Hubbard (2014) who adapted the bias correction method of Karunamuni and Zhang (2008), Zhang, Karunamuni, and Jones (1999) to correct the boundary bias of the two-step value density estimator, which was first proposed by Guerre, Perrigne, and Vuong (2000), of (one-sided) first-price auctions. We generalize their bias correction ideas to correct both boundary and interior biases of bid and value densities which exist in the equilibrium outcome of our double auction model. Furthermore, we establish the uniform convergence rates of our generalized (bias-corrected) density estimators on the whole support for bid densities and on a larger support for value densities.

In view of the preceding results, we consider nonparametric identification and estimation of double auction with bargaining. First, in addition to characterizing all the restrictions on the observables (i.e. bid distributions) imposed by the theoretical double auction model with bargaining, we establish point identification of model primitives (i.e. value distributions) from the observables in the case where all bids are observed. In the case when only transacted bids are observed,[3] we provide a sharp identified set of bidders' value distributions (in Corollary 2).[4] We show that, in the latter case, the conditional distributions of bidders' valuations given positive (conditional) probability of trade are point identified. Second, we propose the (boundary and interior) bias corrected two-step estimators of the buyer's and the seller's value densities. In a double auction setting, we show that our estimators achieve the optimal convergence rate. Third, using Monte Carlo experiments, we show that it is important to implement the bias correction (especially bias correction in the interior of the support) in the two-step estimation of value densities. In particular, we show that, without bias correction, the statistical inference is almost infeasible, not only on the boundaries, but also in the interior.

[2] The asymmetry of bidding strategies arises from the fact that the buyer and the seller have different roles in our double auction model.

[3] In a transaction, a buyer's bid (or offer) must be no lower than seller's bid (or ask).

[4] This result parallels the typical finding that limitations on data observation (such as interval valued data) induce partial identification in nonparametric mean regression and semi-parametric binary regression; see, e.g., Manski and Tamer (2002), Magnac and Maurin (2008), Wan and Xu (2015), among others.

The rest of this paper is organized as follows. In Section 2, we present the sealed-bid double auction model with bargaining and characterize its equilibrium. Section 3 then studies the identification of private value distributions in two different scenarios. In the first scenario, all of the submitted bids can be observed. In contrast, only those bids with successful transactions can be observed in the second scenario. In Section 4, we estimate both the bid and the value densities with bias correction and establish their uniform convergence rates. Section 5 uses Monte Carlo experiments to illustrate the finite sample performance of our estimators. We briefly discuss the extension of our approach to the cases with auction-specific heterogeneity, unobserved heterogeneity, higher order bias reduction, and estimation using transacted bids in Section 6. Section 7 concludes the paper. Appendix A collects the proofs of our main results in the text, while Appendix B presents a supplementary material (the results of which are shown in Appendix C).

2 The k-Double Auction Model

We consider a k-double auction where a single and indivisible object is auctioned between a buyer and a seller. Each of them simultaneously submits a bid. If the buyer's offer is no lower than the seller's ask, a transaction is made at a price of their weighted average, i.e. at a price $p(B,S) = kB + (1-k)S$ where k is a constant in $[0,1]$, B is the buyer's offer, and S is the seller's ask. Otherwise, there is no transaction. The buyer has a value V for the auctioned object, and the seller has a reservation value C. Consequently, the buyer's payoff is $V - p(B,S)$ and the seller's payoff is $p(B,S) - C$ if a trade occurs; their payoffs are both zero otherwise. Each of them does not know her opponent's valuation but only knows that it is drawn from a distribution F_j ($j = C, V$). The distributions F_V, F_C, and the payment rule are all common knowledge between buyer and seller.

We impose the following assumption on the private values and their distributions.

Assumption A. *(i) V and C are independent. (ii) F_V is absolutely continuous on the support $[\underline{v}, \overline{v}] \subset \mathbb{R}_+$ with density f_V. F_C is absolutely continuous on the support $[\underline{c}, \overline{c}] \subset \mathbb{R}_+$ with density f_C.*

Under Assumption A, the seller's private value is independent of the buyer's, and the value distributions are absolutely continuous on bounded supports. Such an assumption has been adopted by most theoretical papers on double auctions with bargaining; see, e.g., Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), Leininger, Linhart, and Radner (1989) and Satterthwaite and Williams (1989).

We also impose the following restriction on the supports of F_V and F_C.

Assumption B. *The supports of F_V and F_C satisfy $\underline{c} < \overline{v}$.*

This assumption requires that the buyer's maximum value must be higher than the seller's minimum cost. It rules out the trivial case of $\overline{v} \leqslant \underline{c}$ in which there is zero probability of trade in any equilibrium. The special cases of such a support condition have been commonly adopted by the theoretical double auction literature; e.g., Myerson and Satterthwaite (1983), Leininger, Linhart, and Radner (1989), and Satterthwaite and Williams (1989).

Denote by $\beta_B : [\underline{v}, \overline{v}] \to \mathbb{R}_+$ and $\beta_S : [\underline{c}, \overline{c}] \to \mathbb{R}_+$ the buyer's and the seller's strategies, respectively. Let $b = \beta_B(v)$ denote the bid of a buyer with realized private value v under strategy β_B. Then, the expected profit of the buyer given the seller's strategy is

$$\pi_B(b,v) = \begin{cases} \int_{\underline{s}}^{b} [v - p(b,s)] \, \mathrm{d}G_S(s) = \int_{\underline{s}}^{b} [v - kb - (1-k)s] \, \mathrm{d}G_S(s), & \text{if } b \geqslant \underline{s}, \\ 0, & \text{if } b < \underline{s}, \end{cases} \tag{2.1}$$

where G_S is the distribution function of the seller's bid and $\underline{s}$ is the lower endpoint of its support. Similarly, let $s = \beta_S(c)$ denote the ask of a seller with realized private reservation value c under strategy β_S. Then, the expected profit of the seller given the buyer's strategy is

$$\pi_S(s,c) = \begin{cases} \int_{s}^{\overline{b}} [p(b,s) - c] \, \mathrm{d}G_B(b) = \int_{s}^{\overline{b}} [kb + (1-k)s - c] \, \mathrm{d}G_B(b), & \text{if } s \leqslant \overline{b}, \\ 0, & \text{if } s > \overline{b}, \end{cases} \tag{2.2}$$

where G_B is the distribution function of the buyer's bid and $\overline{b}$ is the upper endpoint of its support.

We adopt the Bayesian Nash equilibrium (BNE) concept throughout.

Definition 1 (Best response). *A buyer's strategy β_B is a best response to β_S if for any buyer's strategy $\tilde{\beta}_B : [\underline{v}, \overline{v}] \to \mathbb{R}_+$ and each value $v \in [\underline{v}, \overline{v}]$, $\pi_B(\beta_B(v), v) \geqslant \pi_B(\tilde{\beta}_B(v), v)$. The seller's best response is defined in an analogous way.*

Definition 2 (Bayesian Nash equilibrium). *A strategy profile (β_B, β_S) constitutes a Bayesian Nash equilibrium if β_B and β_S are best responses to each other.*

We exclude some irregular equilibria and focus on those which are well-behaved as described in Chatterjee and Samuelson (1983). Precisely, we impose the following restrictions on the equilibrium:

Assumption C (Regular equilibrium). *The equilibrium strategy profile (β_B, β_S) satisfies*

A1. β_B and β_S are continuous on their whole domains;

A2. β_B is continuously differentiable with positive derivative on $[\underline{s}, \overline{v}]$ if $\underline{s} < \overline{v}$; β_S is continuously differentiable with positive derivative on $[\underline{c}, \overline{b}]$ if $\underline{c} < \overline{b}$;

A3. $\beta_B(v) = v$ if $v \leqslant \underline{s}$; $\beta_S(c) = c$ if $c \geqslant \overline{b}$.

We say that an equilibrium satisfying Assumption C is regular. Assumption C basically restricts us to strictly monotone and (piecewise) differentiable strategy equilibria which are quite intuitive in bilateral k-double auctions. As demonstrated by Satterthwaite and Williams (1989, Theorem 3.2), there exist a continuum of regular equilibria when $k \in (0,1)$ and $[\underline{v}, \overline{v}] = [\underline{c}, \overline{c}] = [0,1]$. Following most of the empirical game literature, we adopt the following equilibrium selection mechanism when multiple regular equilibria exist:

Assumption D. *In all observed auctions, the buyers and the sellers play the same regular equilibrium.*

Notice that Assumption D is not restrictive when there is a unique regular equilibrium.

The following lemma characterizes some basic properties of the equilibrium strategy profile.

Lemma 1. *For any equilibrium (β_B, β_S),* *(i) when $v > \underline{s}$, $\beta_B(v) \leqslant v$ with strict inequality if $k > 0$;*

(ii) when $c < \overline{b}$, $\beta_S(c) \geqslant c$ with strict inequality if $k < 1$.

Proof. See Appendix A.1. □

Note that the conclusion of Lemma 1 holds for any BNE (i.e., not only for regular BNE). With condition A3 of Assumption C, it implies that, in regular equilibrium, the buyer will never bid higher than her private value and the seller will never bid lower than her private value. Under the special case of $k = 1/2$, Leininger, Linhart, and Radner (1989) constructed a lemma similar to our Lemma 1.

3 Nonparametric Identification

In this section, we study the nonparametric identification of private value distributions in two cases which differ in the degree of available data. In the first case, researchers can observe both the transacted bids and the bids where no transaction takes place.[5] In the second case, researchers can only observe the transacted bids.

In both cases, we assume that the pricing weight k in the payment rule is known to researchers. Such an assumption is not restrictive because the value of k can be recovered by using some additional information about the transaction price, given that the transacted bids are observed. For example, when the mean transaction price is observed, the parameter k is determined by $k = \frac{\mathbb{E}(P) - \mathbb{E}(S^*)}{\mathbb{E}(B^*) - \mathbb{E}(S^*)}$ since $\mathbb{E}(P) = k\mathbb{E}(B^*) + (1-k)\mathbb{E}(S^*)$ where (B^*, S^*) are the transacted bids. Alternatively, when we observe some quantile of the transaction price, k can be identified by exploiting the property that the price distribution function is continuous and monotone in k (see Appendix B.1 for detailed discussion). In addition, as noted by Section 6.1, the pricing weight k can depend on the heterogeneity when the latter is considered in our framework.

3.1 Case One: All Submitted Bids Being Observed

We first consider the nonparametric identification of the k-double auction model with bargaining when researchers observe the distribution of all submitted bids (including the bids that are not transacted).

As shown in Chatterjee and Samuelson (1983) and Satterthwaite and Williams (1989), a regular equilibrium (β_B, β_S) in a k-double auction with bargaining can be characterized by the following two differential equations for $v \geqslant \underline{s}$ and $c \leqslant \overline{b}$,

$$\beta_B^{-1}(\beta_S(c)) = \beta_S(c) + k\beta_S'(c)\frac{F_C(c)}{f_C(c)}, \tag{3.1}$$

$$\beta_S^{-1}(\beta_B(v)) = \beta_B(v) - (1-k)\beta_B'(v)\frac{1-F_V(v)}{f_V(v)}, \tag{3.2}$$

where $\beta_B^{-1}(\cdot)$ and $\beta_S^{-1}(\cdot)$ are the inverse bidding strategies.[6] For buyer with value $v \geqslant \underline{s}$, the equilibrium bid under strategy β_B is $b = \beta_B(v)$. Let $\tilde{c} = \beta_S^{-1}(b)$. Since strategy β_S is strictly increasing,

[5] We say that a pair of bids (B, S) is transacted if $B \geqslant S$.

[6] When $c = \underline{c}$, (3.1) implies that $\beta_B^{-1}(\underline{s}) = \underline{s}$. Similarly, (3.2) implies that $\beta_S^{-1}(\overline{b}) = \overline{b}$ when $v = \overline{v}$.

$G_S(b) = F_C(\beta_S^{-1}(b)) = F_C(\tilde{c})$. Noting that

$$g_S(b) = \frac{f_C(\beta_S^{-1}(b))}{\beta_S'(\beta_S^{-1}(b))} = \frac{f_C(\tilde{c})}{\beta_S'(\tilde{c})}, \quad v = \beta_B^{-1}(b) = \beta_B^{-1}(\beta_S(\tilde{c})),$$

by (3.1), we have

$$v = b + k\frac{G_S(b)}{g_S(b)}. \tag{3.3}$$

Similarly, for seller with value $c \leqslant \overline{b}$, we have the following condition by (3.2)

$$c = s - (1-k)\frac{1 - G_B(s)}{g_B(s)}. \tag{3.4}$$

Note that (3.3) and (3.4) only hold for $v \geqslant \underline{s}$ and $c \leqslant \overline{b}$. In such a case, we have $\Pr(\beta_B(V) \geqslant \beta_S(C) \mid V = v) > 0$ when $v > \underline{s}$ and $\Pr(\beta_B(V) \geqslant \beta_S(C) \mid C = c) > 0$ when $c < \overline{b}$. In other words, given the private values, both the buyer and the seller expect that trade occurs with positive probability.[7] For the buyer with value $v < \underline{s}$ or the seller with value $c > \overline{b}$, there will be no transaction under strategy profile (β_B, β_S). We define functions $\xi(b, G_S)$ and $\eta(s, G_B)$ as the right-hand sides of (3.3) and (3.4), respectively. That is,

$$\xi(b, G_S) \equiv b + k\frac{G_S(b)}{g_S(b)}, \quad \underline{s} \leqslant b \leqslant \overline{s}, \tag{3.5}$$

$$\eta(s, G_B) \equiv s - (1-k)\frac{1 - G_B(s)}{g_B(s)}, \quad \underline{b} \leqslant s \leqslant \overline{b}. \tag{3.6}$$

By definition, it is straightforward that $\xi(\underline{s}, G_S) = \underline{s}$ and $\eta(\overline{b}, G_B) = \overline{b}$.

We define $\mathscr{P}_{\mathscr{A}}$ as the collection of absolutely continuous probability distributions with support $\mathscr{A}$. Let G denote the joint distribution of (B, S). Here, we restrict ourselves to the regular equilibrium strategies which are strictly increasing and (piecewise) differentiable.

Theorem 1. *Under Assumptions C and D, if $G \in \mathscr{P}_{\mathscr{D}}$ is the joint distribution of regular equilibrium bids (B, S) in a sealed-bid k-double auction with some (F_V, F_C) satisfying Assumptions A and B, then*

C1. The support $\mathscr{D} = [\underline{b}, \overline{b}] \times [\underline{s}, \overline{s}]$ with $\underline{b} \leqslant \underline{s} < \overline{b} \leqslant \overline{s}$;

C2. $G(b, s) = G_B(b) \cdot G_S(s)$ and $G_B \in \mathscr{P}_{[\underline{b}, \overline{b}]}$, $G_S \in \mathscr{P}_{[\underline{s}, \overline{s}]}$;

C3. The function $\xi(\cdot, G_S)$ defined in (3.5) is strictly increasing on $[\underline{s}, \overline{b}]$ and its inverse is differentiable on $[\xi(\underline{s}, G_S), \xi(\overline{b}, G_S)]$;

C4. The function $\eta(\cdot, G_B)$ defined in (3.6) is strictly increasing on $[\underline{s}, \overline{b}]$ and its inverse is differentiable on $[\eta(\underline{s}, G_B), \eta(\overline{b}, G_B)]$;

C5. For any $\overline{b} \leqslant b' \leqslant \overline{s}$ and for any $b \leqslant \overline{b}$ such that $\xi(b, G_S) \geqslant b'$,

$$[\xi(b, G_S) - b']G_S(b') - [\xi(b, G_S) - b]G_S(b) + (1-k)\int_b^{b'} G_S(s)\,\mathrm{d}s \leqslant 0; \tag{3.7}$$

[7] The transaction occurs when $\beta_B(V) \geqslant \beta_S(C)$.

C6. For any $\underline{b} \leqslant s' \leqslant \underline{s}$ *and for any* $s \geqslant \underline{s}$ *such that* $\eta(s, G_B) \leqslant s'$,

$$[s' - \eta(s, G_B)][1 - G_B(s')] - [s - \eta(s, G_B)][1 - G_B(s)] + k \int_{s'}^{s} [1 - G_B(b)]\, \mathrm{d}b \leqslant 0. \tag{3.8}$$

Proof. See Appendix A.2. □

Theorem 1 shows that the theoretical model of a k-double auction with bargaining does impose some restrictions on the joint distribution of observed bids.[8] Together with Theorem 2 which will be shown immediately, these restrictions can be used to establish a formal test of the theory of k-double auction with bargaining. Specifically, condition C1 of Theorem 1 shows that the buyer's minimum (or maximum) bid is not higher than the seller's minimum (or maximum) bid, and the intersection between the buyer's and the seller's bid supports has a non-empty interior. The latter is mainly due to Assumption B about the supports of private value distributions, which implies that there is always positive probability of trade in any regular equilibrium. Condition C2 shows that the buyer's bid is independent of seller's. This independence result is intuitive given that the buyer's value is independent of the seller's. Conditions C3 and C4 say that the functions $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$, which can be regarded as the inverse bidding strategies, are strictly increasing and differentiable on the interval where there is a positive probability of trade. The strict monotonicity property of inverse bidding strategies comes from the fact that the equilibrium strategies are strictly increasing. Conditions C5 and C6 restrict the bid distributions to have small enough probability in the cases where buyer offers less than minimum ask $\underline{s}$ or seller asks more than maximum offer $\overline{b}$.[9]

The following theorem establishes our first identification result.

Theorem 2. *Under Assumptions A to D,* F_V *and* F_C *are point identified from any given* $G \in \mathscr{P}_{\mathscr{D}}$ *satisfying C1–C6.*

Proof. See Appendix A.3. □

Theorem 2 shows that the private value distributions F_V and F_C are point identified from the joint distribution of observed bids. In addition, the inverse bidding strategies $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$ only rely on the knowledge of distribution G. We can hence avoid solving the linked differential equations (3.1) and (3.2) in our identification.

Conditions C5 and C6 are less intuitive, and could be difficult to check in practice. It will be helpful to provide their sufficient conditions which are easy to verify. Our next lemma provides such sufficient conditions.

Lemma 2. *Under Assumptions A to D, conditions C3–C6 are implied by*

C7. The function $\xi(\cdot, G_S)$ *defined in (3.5) is strictly increasing on* $[\underline{s}, \overline{s}]$ *and its inverse is differentiable on* $[\xi(\underline{s}, G_S), \xi(\overline{b}, G_S)]$;

[8] Liu, Vuong, and Xu (2017) characterized similar restrictions imposed by the monotone bayesian Nash equilibrium of theoretical model in the context of binary games with correlated types.

[9] Otherwise, the buyer with very high private value or the seller with very low reservation value will have incentive to deviate from the given equilibrium strategy.

C8. The function $\eta(\cdot, G_B)$ defined in (3.6) is strictly increasing on $[\underline{b}, \overline{b}]$ and its inverse is differentiable on $[\eta(\underline{s}, G_B), \eta(\overline{b}, G_B)]$.

Proof. See Appendix A.4. □

3.2 Case Two: Only Transacted Bids Being Observed

We now discuss the nonparametric identification of the k-double auction model when researchers only observe the transacted bids. This scenario is motivated by the fact that bidders can deviate from truth telling outside the trading region. Thus the unsuccessful bids may not contain much information.

Our identification strategy consists of two key steps. Let G_2 denote the joint distribution of the transacted bids.[10] In the first step, we identify both marginal bid distributions G_B and G_S on $[\underline{s}, \overline{b}]$, from the distribution G_2 of the transacted bids. For any $s \leqslant \overline{b}$, $G_S(s) = \Pr(S \leqslant s | B = \overline{b})$ by the independence between B and S. $\Pr(S \leqslant s | B = \overline{b})$ is identified from the distribution G_2 of the transacted bids, since the transaction is always successful in this case by $S \leqslant s \leqslant \overline{b} = B$. The seller's marginal bid distribution $G_S(\cdot)$ and its density $g_s(\cdot)$ are hence identified on $[\underline{s}, \overline{b}]$. Similarly, the buyer's marginal bid distribution $G_B(\cdot)$ and its density $g_B(\cdot)$ are also identified on $[\underline{s}, \overline{b}]$ by $1 - G_B(b) = \Pr(B > b | S = \underline{s})$ for any $b \geqslant \underline{s}$. In the second step, we recover the corresponding private values for the buyer and the seller by the inverse bidding strategies of (3.5) and (3.6) for the bids on $[\underline{s}, \overline{b}]$.

The above discussion leads to both the rationalization and identification results in case two. We first present the rationalization result as follows.

Corollary 1. *Under Assumptions C and D: If $G_2 \in \mathscr{P}_{\mathscr{D}'}$ is the joint distribution of transacted bids under some regular equilibrium in a sealed-bid k-double auction with (F_V, F_C) satisfying Assumptions A and B, then*

D1. The support $\mathscr{D}' = \left\{(b, s) \,|\, \underline{s} \leqslant s \leqslant b \leqslant \overline{b}\right\}$ with $\underline{s} < \overline{b}$;

D2. For any $\underline{s} \leqslant s' \leqslant s \leqslant b \leqslant b' \leqslant \overline{b}$, the density of G_2 satisfies $g_2(b, s) \cdot g_2(b', s') = g_2(b, s') \cdot g_2(b', s)$;

D3. The function $\xi(\cdot, G_S)$ defined in (3.5) is strictly increasing on $[\underline{s}, \overline{b}]$ and its inverse is differentiable on $[\xi(\underline{s}, G_S), \xi(\overline{b}, G_S)]$;

D4. The function $\eta(\cdot, G_B)$ defined in (3.6) is strictly increasing on $[\underline{s}, \overline{b}]$ and its inverse is differentiable on $[\eta(\underline{s}, G_B), \eta(\overline{b}, G_B)]$.

Proof. See Appendix A.5. □

Corollary 1 shows that the conclusion of Theorem 1 carries over to the transacted bids area, although some (non-transacted) bids cannot be observed now. Specifically, condition D1 says that the support of the distribution of observed (transacted) bids is a triangle in which the buyer's bid is no less than the seller's. Condition D2 means that the multiplication of conditional densities evaluated at (b, s) and (b', s') is the same as the multiplication of conditional densities evaluated at (b, s') and (b', s) as long as these four points are located in the transacted bids area. Such a condition arises mainly due to the independence of private values. Conditions D3 and D4 state that both the buyer's and

[10] Precisely, $G_2(b, s) = \Pr(B \leqslant b, S \leqslant s | \underline{s} \leqslant S \leqslant B \leqslant \overline{b})$.

the seller's inverse bidding strategies are strictly increasing and differentiable on the interval of all possible transacted bids values, namely $[\underline{s}, \overline{b}]$.

We then present the identification result in the following corollary.

Corollary 2. *Suppose that Assumptions A to D hold. For any joint distribution of transacted bids $G_2 \in \mathscr{P}_{\mathscr{D}'}$ satisfying D1–D4, the sharp identified set of value distributions contains all F_V and F_C that satisfy*

E1. $\underline{c} \leqslant \underline{s} < \overline{b} \leqslant \overline{v}$;

E2. For all $(v, c) \in [\underline{s}, \xi(\overline{b}, G_S)] \times [\eta(\underline{s}, G_B), \overline{b}]$,[11]

$$\Pr\left(V \leqslant v \,|\, V \geqslant \underline{s}\right) = \frac{G_B(\xi^{-1}(v, G_S)) - G_B(\underline{s})}{1 - G_B(\underline{s})}, \quad \Pr\left(C \leqslant c \,|\, C \leqslant \overline{b}\right) = \frac{G_S(\eta^{-1}(c, G_B))}{G_S(\overline{b})} \tag{3.9}$$

where $\Pr\left(V \leqslant v \,|\, V \geqslant \underline{s}\right) = \dfrac{F_V(v) - F_V(\underline{s})}{1 - F_V(\underline{s})}$, *and* $\Pr\left(C \leqslant c \,|\, C \leqslant \overline{b}\right) = \dfrac{F_C(c)}{F_C(\overline{b})}$;

E3. For any $b' \geqslant \overline{b}$ and for any $b \leqslant \overline{b}$ such that $\xi(b, G_S) \geqslant b'$,

$$\begin{aligned}&[\xi(b, G_S) - b']F_C(b') - [\xi(b, G_S) - b]F_C(\eta(b, G_B)) \\ &\qquad + (1-k)\left[\int_b^{\overline{b}} F_C(\eta(s, G_B))\,\mathrm{d}s + \int_{\overline{b}}^{b'} F_C(s)\,\mathrm{d}s\right] \leqslant 0;\end{aligned} \tag{3.10}$$

For any $s' \leqslant \underline{s}$ and for any $s \geqslant \underline{s}$ such that $\eta(s, G_B) \leqslant s'$,

$$\begin{aligned}&[s' - \eta(s, G_B)][1 - F_V(s)] - [s - \eta(s, G_B)][1 - F_V(\xi(s, G_S))] \\ &\qquad + k\left\{\int_{s'}^{\underline{s}} [1 - F_V(b)]\,\mathrm{d}b + \int_{\underline{s}}^{s} [1 - F_V(\xi(b, G_S))]\,\mathrm{d}b\right\} \leqslant 0.\end{aligned} \tag{3.11}$$

Proof. See Appendix A.6. □

Corollary 2 gives the identified set of the private value distributions when only the transacted bids are observed. This identified set is actually sharp in the sense that we cannot improve it from the information of observables. Although the private value distributions F_V and F_C are not point identified in this case, the buyer's and the seller's conditional private value distributions are point identified by (3.9) on their value intervals where there is a positive probability of trade.

4 Estimation

Based on the identification strategy, we provide a nonparametric estimation procedure as well as its asymptotic properties when all bids can be observed by the researchers, i.e. in the case one. We will briefly discuss the estimation of case two with transacted bids in Section 6.4. To present the basic ideas, we further assume that all of the observed k-double auctions are homogeneous. Section 6.1 extends our estimation method to allow for auction-specific heterogeneity.

[11] Notice that we have $\xi(\underline{s}, G_S) = \underline{s}$ and $\eta(\overline{b}, G_B) = \overline{b}$ by the definitions of functions ξ and η.

Our estimation procedure extends the two-step estimator proposed by Guerre, Perrigne, and Vuong (2000) for the estimation of sealed-bid first-price auctions: In the first step, a sample of buyers' and sellers' "pseudo private values" is constructed by (3.3) and (3.4), where G_S and G_B are estimated by their empirical distribution functions, and g_S and g_B are estimated by their kernel density estimators with boundary and interior bias correction. In the second step, this sample of pseudo private values is used to nonparametrically estimate the densities of buyers' and sellers' private values with boundary and interior bias correction. Notice that, due to the regular equilibrium assumption, a bidder's private value is equal to her bid (in the first step) if the bidder is a buyer offering less than $\underline{s}$ or if the bidder is a seller asking more than $\overline{b}$.

It is worth pointing out that both boundary and interior bias correction is implemented in all kernel density estimators of our two-step procedure. This is motivated by the fact that the boundary and interior biases are worse in double auctions than in first-price auctions. Specifically, as pointed out by Guerre, Perrigne, and Vuong (2000), the estimators of bid density and private value density suffer from boundary bias (on the two endpoints of each support) in the two-step estimation of first-price auctions, since these two densities are bounded away from zero on finite supports. This issue carries over to the double auction setup, and is made worse by the discontinuity of bid densities in the interior of their supports. The interior discontinuity of bid densities occurs because that the bidding strategies have interior kinks in regular equilibrium. Consequently, the two-step estimator of private value density with boundary and interior bias correction will have a better performance than the one without any bias correction (e.g. the one with sample trimming instead) in finite samples. This is similar to Hickman and Hubbard (2014) who corrected the bias on the boundaries (not in the interior) of the bid and value densities, and is confirmed by our Monte Carlo experiments in Section 5 as well.

We adapt the boundary correction technique proposed by Zhang, Karunamuni, and Jones (1999) and Karunamuni and Zhang (2008) to our double auction setup, and follow them to focus on the case of continuously differentiable private value density (and hence twice continuously differentiable bid density by Lemma 3). The case of smoother private value densities is discussed in Section 6.3.

4.1 Definition of the Estimator

To clarify our idea, we consider n homogeneous k-double auctions. In each auction $i = 1, 2, \ldots, n$, there is one buyer with private value V_i and one seller with private value C_i. We observe a sample that consists of all of the buyers' bids $\{B_1, B_2, \ldots, B_n\}$ and all of sellers' bids $\{S_1, S_2, \ldots, S_n\}$. Let $\hat{\underline{b}}$ and $\hat{\overline{b}}$ ($\hat{\underline{s}}$ and $\hat{\overline{s}}$) be the minimum and maximum of the buyers' (sellers') n observed bids.

Our estimation proceeds as follows: In the first step, we use the observed sample of all bids to estimate the distribution and density functions of the buyers' and sellers' bids by their empirical distribution functions and (boundary and interior) bias corrected kernel density estimators, respectively, i.e. by

$$\hat{G}_B(b) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(B_i \leqslant b), \quad \hat{G}_S(s) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(S_i \leqslant s),$$

and kernel density estimators $\hat{g}_B$ and $\hat{g}_S$ which are estimated on $[\hat{\underline{s}}, \hat{\overline{b}}]$ as shown in (B.5) of Appendix B.2. Specifically, the estimator of the buyers' bid density $\hat{g}_B$ uses kernel function K_B, primary bandwidth h_B, secondary bandwidth h'_B and coefficient $A = A_B$, while the estimator of the sellers' bid density $\hat{g}_S$

uses K_S, h_S, h'_S and $A = A_S$. We then define the buyer's pseudo private value $\hat{V}_i$ corresponding to B_i and the seller's pseudo private value $\hat{C}_i$ corresponding to S_i, respectively, as

$$\hat{V}_i = \begin{cases} B_i + k\dfrac{\hat{G}_S(B_i)}{\hat{g}_S(B_i)} & \text{if } B_i \geqslant \hat{\underline{s}}, \\ B_i & \text{otherwise,} \end{cases} \qquad \hat{C}_i = \begin{cases} S_i - (1-k)\dfrac{1-\hat{G}_B(S_i)}{\hat{g}_B(S_i)} & \text{if } S_i \leqslant \hat{\overline{b}}, \\ S_i & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\hat{G}_B(\cdot), \hat{G}_S(\cdot), \hat{g}_B(\cdot)$, and $\hat{g}_S(\cdot)$ are the empirical distribution functions and bias-corrected kernel density estimators defined earlier. Note that we have $V_i = B_i$ (resp. $C_i = S_i$) when $B_i < \underline{s}$ (resp. $S_i > \overline{b}$) in regular equilibrium.

In the second step, we use the pseudo private value samples, $\{\hat{V}_1, \ldots, \hat{V}_n\}$ and $\{\hat{C}_1, \ldots, \hat{C}_n\}$, to estimate the buyers' and sellers' respective value densities. Specifically, the estimator of the buyers' value density $\hat{f}_V$ is obtained by applying the bias correction approach in (B.5) to the sample of the buyers' pseudo private values on $[\hat{\underline{v}}, \hat{\overline{v}}]$, where $\hat{\underline{v}}$ and $\hat{\overline{v}}$ are respectively the minimum and maximum of the buyers' pseudo private values, with kernel function K_V, primary bandwidth h_V, secondary bandwidth h'_V, and coefficient $A = A_V$. Similarly, we get the estimator of the sellers' value density $\hat{f}_C$ on interval $[\hat{\underline{c}}, \hat{\overline{c}}]$ by the sample of the sellers' pseudo private values with kernel function K_C, primary bandwidth h_C, secondary bandwidth h'_C, and coefficient $A = A_C$.

4.2 Asymptotic Properties

The next assumption concerns the generating process of buyers' and sellers' private values $(V_i, C_i), i = 1, \ldots, n$.

Assumption E. *V_i, $i = 1, 2, \ldots, n$, are independently and identically distributed as F_V with density f_V; C_i, $i = 1, 2, \ldots, n$, are independently and identically distributed as F_C with density f_C.*

This assumes that the bidders' private values are independent across auctions. In addition, we impose a smoothness condition on the latent value distributions as follows:

Assumption F. *F_V and F_C are twice continuously differentiable on $[\underline{v}, \overline{v}]$ and $[\underline{c}, \overline{c}]$, respectively. In addition, $f_V(v) \geqslant \alpha_V > 0$ for all $v \in [\underline{v}, \overline{v}]$; $f_C(c) \geqslant \alpha_C > 0$ for all $c \in [\underline{c}, \overline{c}]$.*

Assumption F requires that, on compact supports, the latent value distributions are twice continuously differentiable and their density functions are bounded away from zero. As shown in the following lemma, this assumption implies that the generated equilibrium bid distributions will also satisfy a similar smoothness condition.

Lemma 3. *Given Assumption F, the distributions of regular equilibrium bids G_B and G_S satisfy:*

(i) for any $b \in [\underline{b}, \overline{b}]$ and any $s \in [\underline{s}, \overline{s}]$, $g_B(b) \geqslant \alpha_B > 0$, $g_S(s) \geqslant \alpha_S > 0$;

(ii) G_B and G_S are twice continuously differentiable on $[\underline{s}, \overline{b}]$;

(iii) g_B and g_S are also twice continuously differentiable on $[\underline{s}, \overline{b}]$.

Proof. See Appendix A.7. □

The striking feature of Lemma 3 is part (iii). It shows that the bid densities are smoother than their corresponding latent value densities. A similar result is obtained by Guerre, Perrigne, and Vuong (2000) in first-price auctions.

We turn to the choice of kernels in the following assumption.

Assumption G. *K_B, K_S, K_V and K_C are symmetric second order kernels with support $[-1,1]$ and have continuous bounded second order derivatives.*

We then give conditions on the choice of bandwidths and other tuning parameters.

Assumption H. *The bandwidths h_B, h_S, h_V, h_C are of the form:*

$$h_B = \lambda_B\big(\log n/n\big)^{1/5}, \quad h_S = \lambda_S\big(\log n/n\big)^{1/5}, \quad h_V = \lambda_V\big(\log n/n\big)^{1/5}, \quad h_C = \lambda_C\big(\log n/n\big)^{1/5},$$

where the λ's are positive constants. The parameters $A_B, A_S, A_V, A_C > 1/3$ and the secondary bandwidths are of the form: $h'_B = \tau_B n^{-1/5}$, $h'_S = \tau_S n^{-1/5}$, $h'_V = \tau_V n^{-1/5}$, $h'_C = \tau_C n^{-1/5}$, where the τ's are positive constants.

To implement the bias correction technique, we adopt Assumption H to choose all primary bandwidths of order $(\log n/n)^{1/5}$ and the secondary bandwidths h'_B, h'_S, h'_V, and h'_C of order $n^{-1/5}$.[12]

Our main estimation result establishes the uniform consistency (with rates of convergence) of the two-step estimators of value densities. It is built on the following lemma which shows the uniform consistency (with rates of convergence) of (i) the first-step nonparametric estimators of the bid densities and (ii) the pseudo private values $\hat{V}_i$ and $\hat{C}_i$.

Lemma 4. *Suppose that Assumptions E to H hold, then*

(i) $\sup_{b\in[\underline{b},\bar{b}]} |\hat{g}_B(b) - g_B(b)| = O_p\big((\log n/n)^{2/5}\big), \quad \sup_{s\in[\underline{s},\bar{s}]} |\hat{g}_S(s) - g_S(s)| = O_p\big((\log n/n)^{2/5}\big).$

(ii) $\sup_i |\hat{V}_i - V_i| = O_p\big((\log n/n)^{2/5}\big), \quad \sup_i |\hat{C}_i - C_i| = O_p\big((\log n/n)^{2/5}\big).$

Proof. See Appendix A.8. □

Lemma 4 first shows that, after bias correction, the kernel density estimators of the bid distributions uniformly converge in probability to the true densities at a rate of $(\log n/n)^{2/5}$ on their entire supports. It also shows that all pseudo private values converge uniformly in probability to the true private values at the same rate. Without boundary and interior bias correction, the uniform convergence of bid density estimators and pseudo private values holds only on an interior closed subset (excluding boundaries) of bid support.

We now give our main result of the estimation section.

Theorem 3. *Under Assumptions E to H, for any (fixed) closed inner subsets $\mathscr{C}_V$ of $[\underline{v},\bar{v}]$ and $\mathscr{C}_C$ of $[\underline{c},\bar{c}]$,* [13]

$$\sup_{v\in\mathscr{C}_V} |\hat{f}_V(v) - f_V(v)| = O_p\big((\log n/n)^{1/5}\big), \quad \sup_{c\in\mathscr{C}_C} |\hat{f}_C(c) - f_C(c)| = O_p\big((\log n/n)^{1/5}\big).$$

[12] Such choices of secondary bandwidths minimize the mean squared errors of estimating d's in the transform functions for bias correction.

[13] We call closed set $\mathscr{A}' \subseteq \mathscr{A}$ a closed inner subset of $\mathscr{A}$ if $\mathscr{A}'$ is also a subset of the interior of $\mathscr{A}$.

Proof. See Appendix A.9. $\square$

Theorem 3 shows that our (bias corrected) two-step estimators of private value densities converge uniformly to their true densities at a rate of $(\log n/n)^{1/5}$ on any closed inner subset of value support. This convergence rate is optimal in first-price auctions (Guerre, Perrigne, and Vuong, 2000). Without bias correction, the usual two-step estimators of private value densities have the same convergence rate as $(\log n/n)^{1/5}$ only on any close inner subset excluding $\underline{s}$ (or $\overline{b}$). Consequently, we expect that, in comparison to the two-step estimator without bias correction, the one with bias correction will have better finite sample performance close to $\underline{s}$ for the buyers' value density estimator and close to $\overline{b}$ for the sellers'. This is confirmed by our Monte Carlo experiments in the next section. Nevertheless, Theorem 3 does not say anything about the uniform convergence rate on the entire support. The main difficulty comes from the low accuracy in estimation of the boundary points $\underline{v}$, $\overline{v}$, $\underline{c}$ and $\overline{c}$, since they are estimated from the pseudo private values which converge to their true values at a nonparametric rate.

With smoother value densities, the uniform convergence rate of the two-step value density estimators with and without bias correction can be improved. Such an extension is briefly discussed in Section 6.3 for bias corrected estimators, and in Appendix B.3 of supplementary material for estimators without bias correction.

5 Monte Carlo Experiments

To study the finite sample performance of our two-step estimation procedure, we conduct Monte Carlo experiments. We consider two cases of buyers' and sellers' true value distributions and pricing weights. In the first case, both buyers' and sellers' private values are uniformly distributed on $[0,1]$. The bidding strategies of the buyer and the seller are given by

$$\beta_B(v) = \begin{cases} \dfrac{v}{1+k} + \dfrac{k(1-k)}{2(1+k)}, & \text{if } \dfrac{1-k}{2} \leqslant v \leqslant 1, \\ v, & \text{if } 0 \leqslant v < \dfrac{1-k}{2}; \end{cases} \quad \beta_S(c) = \begin{cases} \dfrac{c}{2-k} + \dfrac{1-k}{2}, & \text{if } 0 \leqslant c \leqslant \dfrac{2-k}{2}, \\ c, & \text{if } \dfrac{2-k}{2} < c \leqslant 1, \end{cases}$$

where k is the pricing weight. Moreover, we set the pricing weight $k = 1/2$ so that the buyer and the seller have equal bargaining power in determining the transaction price. This case has been frequently studied in the theoretical literature (e.g., Chatterjee and Samuelson, 1983). In the second case, we allow asymmetry between buyers' and sellers' value distributions, and asymmetry between their pricing weights. Specifically, we set the pricing weight to $k = 3/4$, and the true densities of buyers' and sellers' private value distributions to be:

$$f_V(v) = \frac{(8v+12)\sqrt{16v^2 - 128v + 553} - 32v^2 + 80v - 105}{(7\sqrt{553} - 31)\sqrt{16v^2 - 128v + 553}},$$
$$f_C(c) = \frac{1}{511 + \sqrt{73} - 1076e^{-3/4}} \left[4 - \frac{8c}{9} + \frac{9+16c}{\sqrt{81+16c^2}} - \frac{2}{9}\sqrt{81+16c^2} + \mathbb{1}(c \geqslant 3)\frac{(c-3)^3}{3}e^{\frac{3-c}{4}}\right],$$

with identical supports, $[\underline{v}, \overline{v}] = [\underline{c}, \overline{c}] = [0, 6]$.[14] In this case, it can be verified that the buyer's and the seller's bidding strategies given by[15]

$$\beta_B(v) = \begin{cases} v, & \text{if } 0 \leqslant v < 1, \\ \dfrac{4v + 28 - \sqrt{16v^2 - 128v + 553}}{11}, & \text{if } 1 \leqslant v \leqslant 6; \end{cases}$$

$$\beta_S(c) = \begin{cases} \dfrac{4c + \sqrt{16c^2 + 81}}{9}, & \text{if } 0 \leqslant c \leqslant 3, \\ c, & \text{if } 3 < c \leqslant 6, \end{cases}$$

form a regular equilibrium. Figure 1 plots the true value densities, the equilibrium bidding strategies, and the induced bid densities in the second case.

Our Monte Carlo experiment consists of 5000 replications for each case. In each replication, we first randomly generate n buyers' and n sellers' private values from their true value distributions. We then compute the corresponding bids according to the true bidding strategies. Next, we apply our bias-corrected two-step estimation procedure to the generated sample of bids for each replication. In the first step, we estimate the distribution functions and densities of buyers' and sellers' bids using the empirical distribution functions and bias-corrected kernel density estimators, respectively. We then use (4.1) to obtain the buyers' and the sellers' pseudo private values. In the second step, we use the sample of buyers' and sellers' pseudo private values to estimate buyers' and sellers' value densities by their bias-corrected kernel density estimators.

To satisfy Assumption G on the kernels,[16] we choose the triweight kernel for all of $K_B(\cdot)$, $K_S(\cdot)$, $K_V(\cdot)$, and $K_C(\cdot)$, i.e. $K_B(u) = K_S(u) = K_V(u) = K_C(u) = (35/32)(1-u^2)^3 \cdot \mathbb{1}(-1 \leqslant u \leqslant 1)$. We then choose the primary bandwidths h_B, h_S, h_V and h_C according to the rule of optimal global bandwidth (see Silverman, 1986) as

$$h_j = \min\left(n^{-\frac{1}{5}} \hat{\sigma}_j \left[\frac{8\sqrt{\pi} \int_{-1}^{1} K_j^2(u)\,\mathrm{d}u}{3\left(\int_{-1}^{1} u^2 K_j(u)\,\mathrm{d}u\right)^2} \right]^{\frac{1}{5}}, \frac{\hat{r}_j}{2} \right), \quad j = B, S, V, C,$$

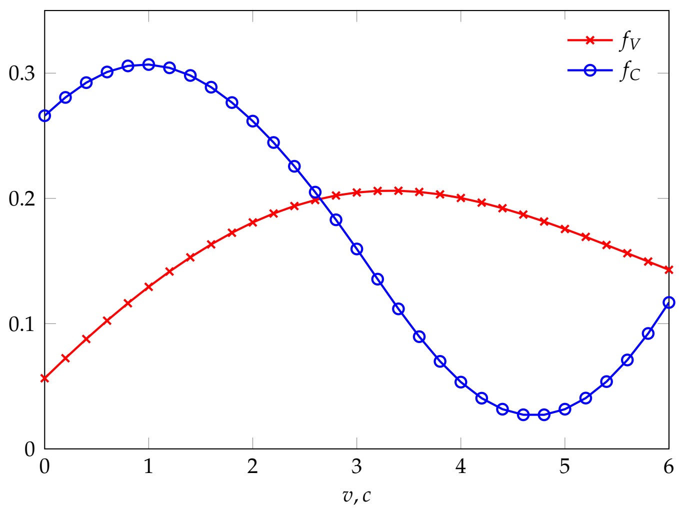
where n is the sample size of the observed bids, $\hat{\sigma}_j$ is the estimated standard deviation of observed bids for $j = B, S$ or pseudo private values for $j = V, C$, $K_j(\cdot)$ is the kernel function, and $\hat{r}_j$ is the length of the interval on which the corresponding bid or value density is estimated. In addition, the parameters of bias correction are chosen as follows: all of the coefficients A_B, A_S, A_V and A_C are set at

[14] As a matter of fact, we also add some curvature to the true value densities $f_V(\cdot)$ and $f_C(\cdot)$ in this case.

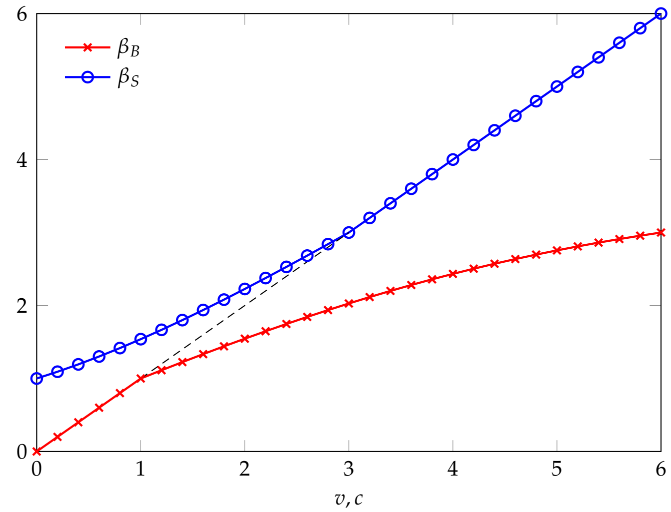
[15] It can also be verified that the corresponding bid densities are

$$g_B(b) = \begin{cases} f_V(b), & \text{if } 0 \leqslant b < 1, \\ \dfrac{121b}{28\sqrt{553} - 124}, & \text{if } 1 \leqslant b \leqslant 3, \\ 0, & \text{otherwise;} \end{cases} \qquad g_S(s) = \begin{cases} \dfrac{36 - 9s}{2044 + 4\sqrt{73} - 4304e^{-3/4}}, & \text{if } 1 \leqslant s \leqslant 3, \\ f_C(s), & \text{if } 3 < s \leqslant 6, \\ 0, & \text{otherwise.} \end{cases}$$

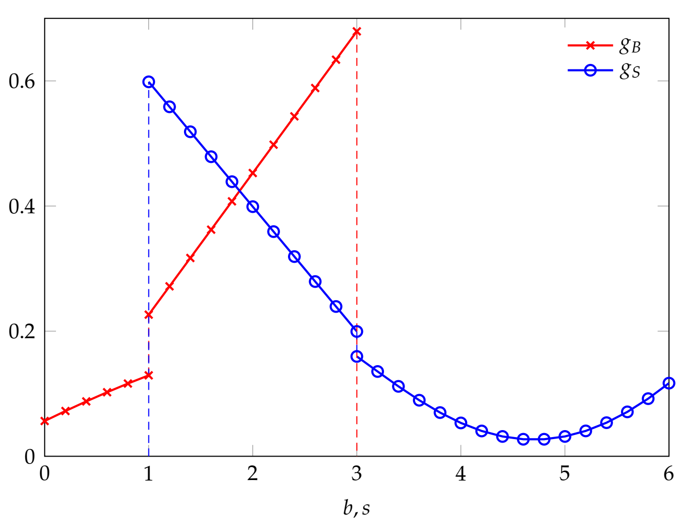
[16] Notice that, in both cases, the private value densities $f_V(\cdot)$ and $f_C(\cdot)$ are continuously twice differentiable on the entire support.



(a) Buyers' and sellers' private value densities



(b) Equilibrium bidding strategies



(c) Density functions of induced bids

Figure 1: True private value densities, equilibrium bidding strategies and bid densities in the second experiment

0.65; each of the secondary bandwidths is equal to its counterpart among the primary bandwidths,[17] i.e. $h'_j = h_j$ for $j = B, S, V, C$.

Our Monte Carlo results for the first case are summarized in Figure 2. It shows the two-step estimates of value densities with and without bias correction under the sample sizes of $n = 200$ and $n = 1000$, when both buyers' and sellers' private values are uniformly distributed on $[0, 1]$. The true value densities are displayed in solid lines. For each value of $v \in [0, 1]$ (or $c \in [0, 1]$), we plot the mean of the estimates with a dashed line, and the 5th and 95th percentiles with dotted lines. The latter gives the (pointwise) 90% confidence interval for $f_V(v)$ (or $f_C(c)$). Figure 2 shows that our bias-corrected two-step density estimates behave well. First, the true curves fall within their corresponding confidence bands. Second, the mean of the estimates for each density closely matches the true curve. Third, as sample size increases, both the bias and variance of the estimates decrease. Figure 2 also shows that bias correction plays an important role in estimating the value densities in double auctions with bargaining. As shown by Figures 2c, 2d, 2g and 2h, the standard kernel density estimator (without bias correction) has large bias not only at the boundaries but also in an interior area. When the sample size n increases, this bias will not diminish, although the variance will shrink. The appearance of bias in the interior shows that bias correction is necessary to estimate value densities in double auctions with bargaining.

Figure 3 reports the simulation results of the second case under the sample sizes of $n = 200$ and $n = 1000$. Similarly, the true densities, means, and 5th/95th percentiles are respectively displayed in solid lines, dashed lines, and dotted lines. It shows that, with some curvature in the value densities and asymmetry between buyers and sellers, the conclusions in Figure 2 still hold; that is, (i) the bias-corrected two-step density estimates perform well, and (ii) bias correction plays an important role for estimating the value densities in our double auction model.

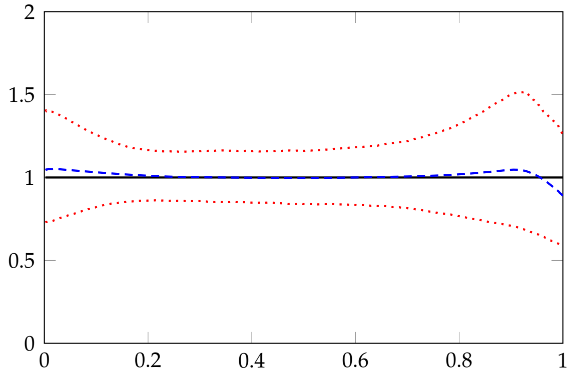
6 Extensions

6.1 Auction-Specific Heterogeneity

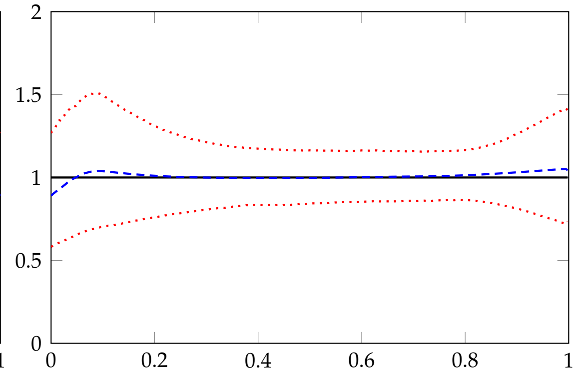
We now briefly discuss how to generalize our identification and estimation approach to allow for auction-specific heterogeneity.[18] Let $X \in \mathbb{R}^d$ be a random vector that characterizes the heterogeneity of auctions. For auctions with $X = x$, let $F_{V|X}(\cdot|x)$ and $F_{C|X}(\cdot|x)$ be the buyers' and sellers' private value distributions, and $G_{B|X}(\cdot|x)$ and $G_{S|X}(\cdot|x)$ be their respective bid distribution functions with densities $g_{B|X}(\cdot|x)$ and $g_{S|X}(\cdot|x)$. Let all of our previous assumptions hold for every x in the support of X wherever it applies. The buyer's and the seller's inverse bidding functions in an auction with

[17] We tried other values of coefficients A_j and secondary bandwidths h'_j, $j = B, S, V, C$, in our experiments, but found that, as long as Assumption H holds, the estimates of both buyers' and sellers' value densities are almost the same for different values of A_j and h'_j.

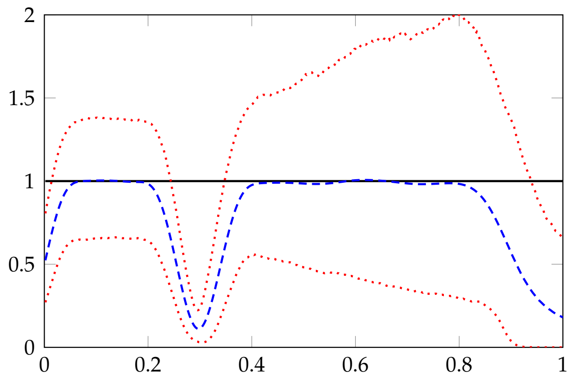
[18] The existence of auction-specific heterogeneity allows for correlation between the buyer's and the seller's private values. Such correlation, however, exists only through the auction-specific heterogeneity.



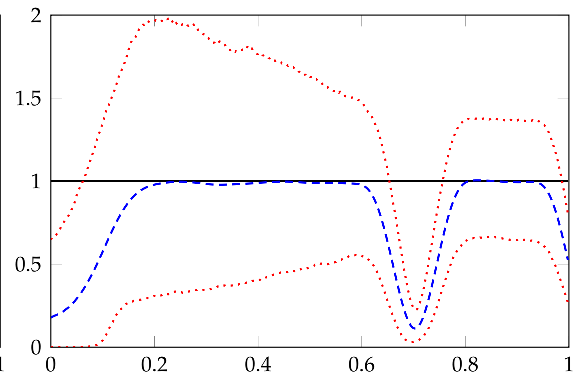
(a) $\hat{f}_V(v)$, $n = 200$



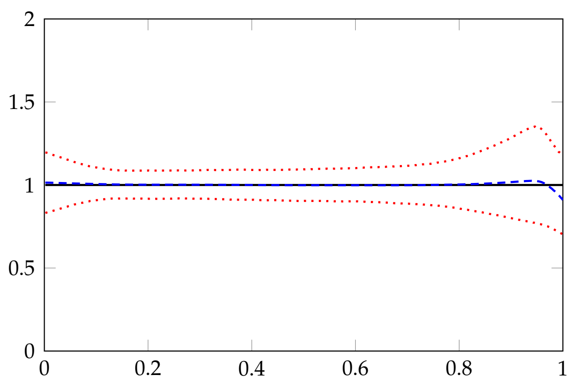
(b) $\hat{f}_C(c)$, $n = 200$



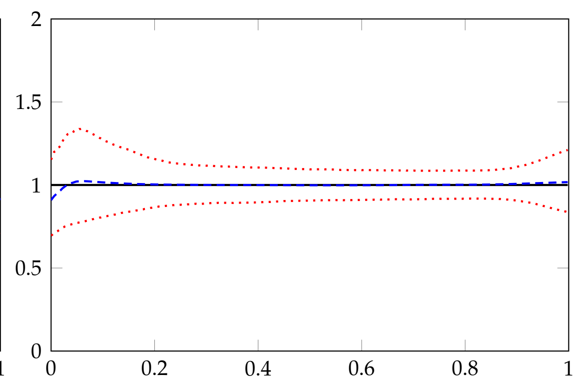
(c) $\hat{f}_V(v)$, $n = 200$, no bias correction



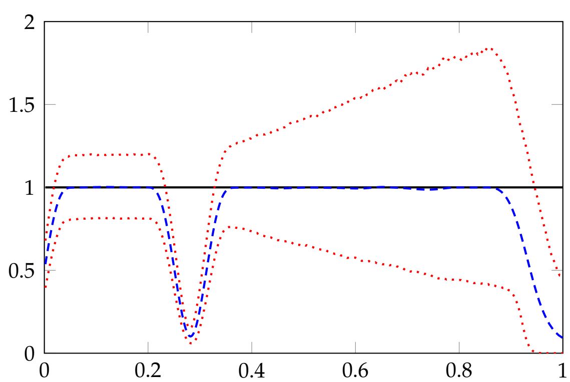
(d) $\hat{f}_C(c)$, $n = 200$, no bias correction



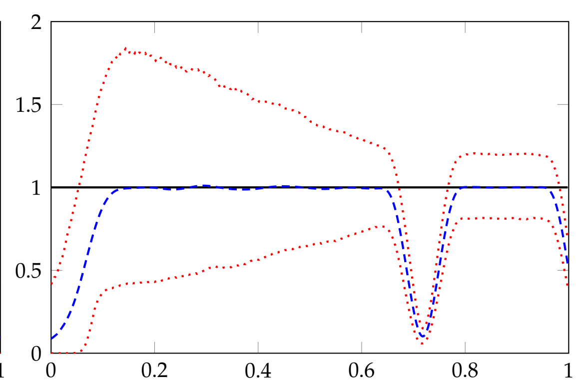
(e) $\hat{f}_V(v)$, $n = 1000$



(f) $\hat{f}_C(c)$, $n = 1000$

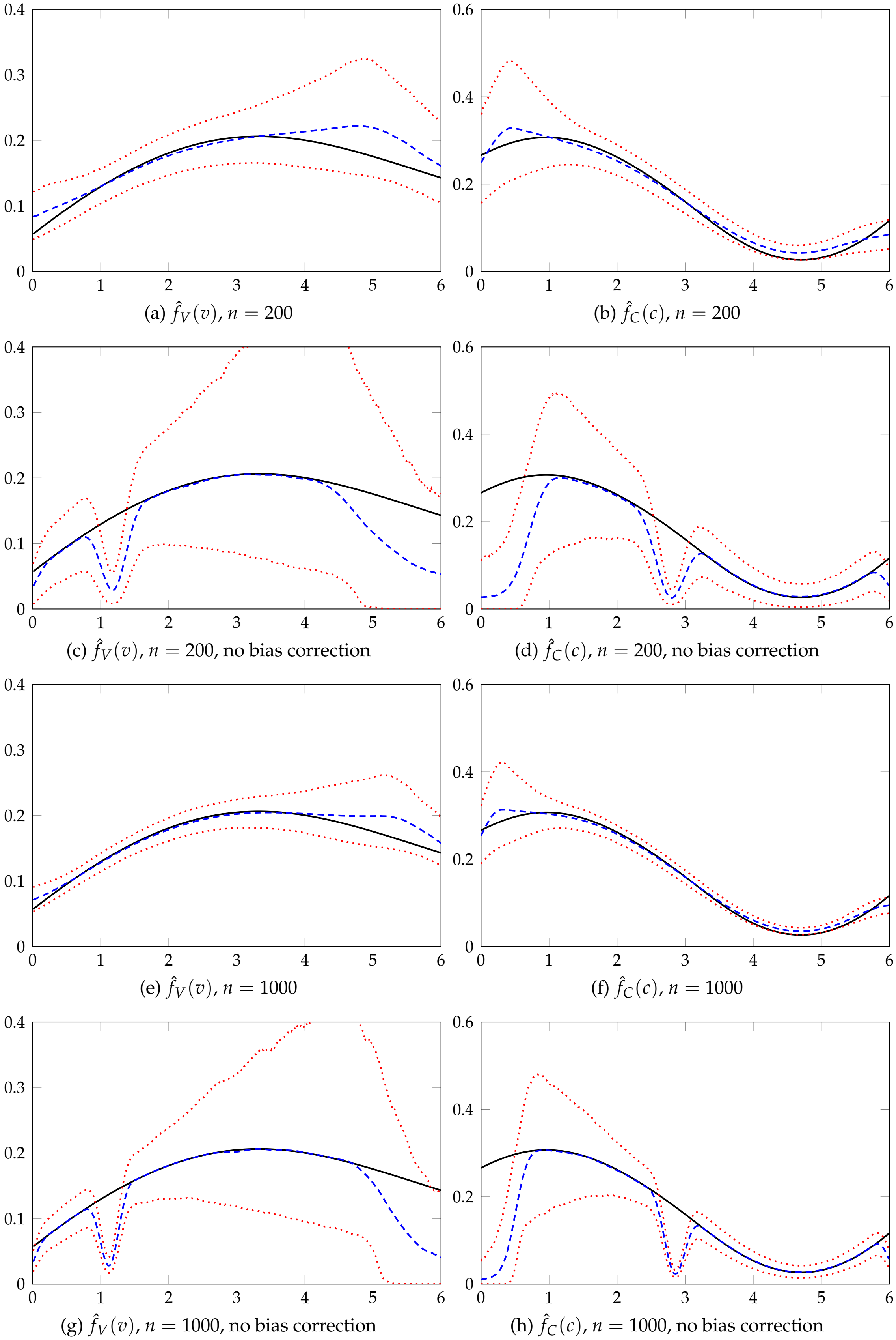


(g) $\hat{f}_V(v)$, $n = 1000$, no bias correction



(h) $\hat{f}_C(c)$, $n = 1000$, no bias correction

Figure 2: True and estimated densities of private values. $V_i \sim U[0, 1]$, $C_i \sim U[0, 1]$.



(a) $\hat{f}_V(v)$, $n = 200$

(b) $\hat{f}_C(c)$, $n = 200$

(c) $\hat{f}_V(v)$, $n = 200$, no bias correction

(d) $\hat{f}_C(c)$, $n = 200$, no bias correction

(e) $\hat{f}_V(v)$, $n = 1000$

(f) $\hat{f}_C(c)$, $n = 1000$

(g) $\hat{f}_V(v)$, $n = 1000$, no bias correction

(h) $\hat{f}_C(c)$, $n = 1000$, no bias correction

Figure 3: True and estimated densities of private values under asymmetry.

characteristic $X = x$ are, respectively,

$$v = \begin{cases} b + k(x) \cdot \dfrac{G_{S|X}(b \mid x)}{g_{S|X}(b \mid x)}, & \text{if } b \geqslant \underline{s}(x), \\ b, & \text{otherwise,} \end{cases} \qquad c = \begin{cases} s - (1 - k(x)) \cdot \dfrac{1 - G_{B|X}(s \mid x)}{g_{B|X}(s \mid x)}, & \text{if } s \leqslant \overline{b}(x), \\ s, & \text{otherwise,} \end{cases} \tag{6.1}$$

where $\underline{s}(x)$ is the lower bound of the support of $G_{S|X}(\cdot \mid x)$, $\overline{b}(x)$ is the upper bound of the support of $G_{B|X}(\cdot \mid x)$. Note that the weight $k(x)$ depends on the realization of heterogeneity X in this case.

We can then generalize most of our identification and estimation results to auctions with heterogeneity. Specifically, our identification and model restrictions results (Theorems 1 and 2, Lemma 2, and Corollaries 1 and 2) still hold as long as the value and bid distributions are simply replaced by the corresponding conditional distributions given X and all relevant conditions hold for every realization of X.

For estimation, our two-step procedure can be generalized to incorporate auction-specific heterogeneity. In the first step, for each auction, we use (6.1) to recover both the buyers' and the sellers' pseudo private values. Notice that, in (6.1), the estimation of conditional bid densities $g_{S|X}$ and $g_{B|X}$ needs to first recover the joint densities g_{SX} and g_{BX} of the bids and the covariates (as well as the marginal density f_X of the covariates), since $g_{S|X}(s \mid x) = g_{SX}(s, x)/f_X(x)$ and $g_{B|X}(b \mid x) = g_{BX}(b, x)/f_X(x)$. In the second step, we use the covariate data $\{X_1, \ldots, X_n\}$ and pseudo private values recovered previously to estimate the conditional value densities $f_{C|X}$ and $f_{V|X}$. Again, this needs the estimation of joint densities of valuation and covariates f_{CX} and f_{VX}. It is then possible to extend our estimation results in Section 4 to this new two-step estimator. However, the new estimator will suffer the "curse of dimensionality" with the introduction of auction-specific heterogeneity $X \in \mathbb{R}^d$. Moreover, for $d \geqslant 1$, the (interior and boundary) bias correction in kernel estimation of bid densities g_{SX} and g_{BX} will be an issue in a multi-dimensional scenario.[19] This issue is challenging, in that, to our knowledge, little is known in the existing literature regarding the boundary bias correction of kernel density estimators in a multi-dimensional setting.

6.2 Unobserved heterogeneity

Our framework can incorporate an auction-level unobserved heterogeneity. We omit the conditioning on the observed covariates X to simplify our discussion. Let $\tilde{X}$ represent the unobserved heterogeneity, namely $\tilde{X}$ is observed by all bidders but unobserved by the researchers. All bidders can hence condition on it when bidding.

We consider that the buyer's value (resp. seller's cost) has a multiplicatively separable form of $V = \tilde{X} \cdot \epsilon$ (resp. $C = \tilde{X} \cdot \delta$) where ϵ and δ are private information to buyer and seller, respectively. Let $\beta_{B,\tilde{x}}(\cdot)$ and $\beta_{S,\tilde{x}}(\cdot)$ be the buyer's and seller's bidding strategies under $\tilde{X} = \tilde{x}$. Suppose that $\beta_{B,1}(\cdot)$ and $\beta_{S,1}(\cdot)$ are the equilibrium bidding strategies under $\tilde{X} = 1$, i.e. they satisfy the first order conditions (3.1) and (3.2) when $\tilde{X} = 1$. It can be verified that $\beta_{B,\tilde{x}}(\cdot) = \tilde{x} \cdot \beta_{B,1}(\cdot/\tilde{x})$ and $\beta_{S,\tilde{x}}(\cdot) = \tilde{x} \cdot \beta_{S,1}(\cdot/\tilde{x})$ also satisfy (3.1) and (3.2) and are therefore equilibrium bidding strategies when $\tilde{X} = \tilde{x}$.

[19] Notice that the supports of S and B are finite. In addition, the bid densities can have discontinuity points in the interior of the supports (see Figure 1c).

Let $B = \beta_{B,\tilde{X}}(V)$ and $S = \beta_{S,\tilde{X}}(C)$. The above discussion yields $B = \tilde{X} \cdot \beta_{B,1}(V/\tilde{X}) = \tilde{X} \cdot \beta_{B,1}(\epsilon)$ and $S = \tilde{X} \cdot \beta_{S,1}(C/\tilde{X}) = \tilde{X} \cdot \beta_{S,1}(\delta)$. Taking logarithms gives

$$\log(B) = \log(\tilde{X}) + \log(\beta_{B,1}(\epsilon))$$
$$\log(S) = \log(\tilde{X}) + \log(\beta_{S,1}(\delta)).$$

We can then apply the deconvolution approach of Krasnokutskaya (2011) to identify the distributions of $\tilde{X}$, ϵ, and δ under some scale normalizations.

A similar strategy can be applied to identify the distributions of unobserved heterogeneity $\tilde{X}$ and private information ϵ and δ when the buyer's value (resp. seller's cost) has an additively separable form of $V = \tilde{X} + \epsilon$ (resp. $C = \tilde{X} + \delta$).

6.3 Higher order bias reduction

When the value density function is smoother, we can also have higher order boundary (and interior) bias reduction at the cost of more tedious calculations. Due to space limitations, we only illustrate the idea of achieving higher order bias reduction here.

To achieve higher order boundary (and interior) bias reduction, we need to specify both a higher order kernel and a proper functional form for the data transformation. For demonstration purposes, suppose that $\{X_1, X_2, \ldots, X_n\}$ is a random sample drawn from a distribution with a density function $f(\cdot)$ admitting up to $R+1$ continuous bounded derivatives on a support of $[0, \overline{x}]$. To simplify the analysis, we further assume that the density $f(\cdot)$ has a discontinuity point only at 0, i.e. we assume $\lim_{x \to \overline{x}^-} f(x) = 0$. Denote the transformation function by $\gamma(\cdot)$.[20] The (boundary-corrected) kernel density estimator of $f(\cdot)$ with a generalized reflection is given by

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} \left[K\left(\frac{x - X_i}{h}\right) + K\left(\frac{x + \gamma(X_i)}{h}\right) \right],$$

where $K(\cdot)$ is a kernel function on support $[-1, 1]$, and h is a bandwidth parameter. Let $\omega(\cdot) = f(\gamma^{-1}(\cdot))/\gamma'(\gamma^{-1}(\cdot))$ with $\gamma(\cdot)$ being strictly increasing on $[0, +\infty)$ and $(R+1)$-times continuously differentiable. Then, for $x = \rho h$ with $0 \leqslant \rho \leqslant 1$, the bias of $\hat{f}$ at x can be obtained as

$$E\hat{f}(x) - f(x) = [\omega(0) - f(0)] \int_{\rho}^{1} K(t)\, \mathrm{d}t + \sum_{j=1}^{R} \frac{W_j}{j!} h^j + O\left(h^{R+1}\right), \tag{6.2}$$

where

$$W_j = f^{(j)}(0) \left[\sum_{l=1}^{j} \binom{j}{l} (-1)^l \rho^{j-l} \int_{-1}^{1} t^l K(t)\, \mathrm{d}t \right] + \left[\omega^{(j)}(0) - (-1)^j f^{(j)}(0) \right] \int_{\rho}^{1} (t-\rho)^j K(t)\, \mathrm{d}t.$$

Consequently, if we choose a kernel $K(\cdot)$ of order $(R+1)$ and a transformation function $\gamma(\cdot)$ such that (i) $\omega(0) = f(0)$, (ii) $\omega^{(j)}(0) = (-1)^j f^{(j)}(0)$ for all $j = 1, 2, \ldots, R$, (iii) $\gamma'(\cdot) > 0$ on $[0, +\infty)$,

[20] In Section 4, we follow Zhang, Karunamuni, and Jones (1999) and Karunamuni and Zhang (2008) and employ a cubic transformation function of $\gamma(u) = u + d \cdot u^2 + A \cdot d^2 \cdot u^3$ where d is the derivative of log-density at the boundary point.

and (iv) $(R+1)$-th derivative of $\gamma(\cdot)$ exists, then the boundary bias $E\hat{f}(x) - f(x) = O(h^{R+1})$ for any $x = \rho h$ with $0 \leqslant \rho \leqslant 1$. To see this, condition (i) eliminates the first term on the right-hand side of (6.2), and condition (ii) together with $(R+1)$-th order kernel $K(\cdot)$ implies $W_j = 0$ for all $j = 1, \ldots, R$ which makes the second term on the right-hand side of (6.2) zero. With the bias of order $O(h^{R+1})$ on the boundary, the kernel density estimator $\hat{f}(\cdot)$ with a generalized reflection then converges uniformly to the true density function $f(\cdot)$ at a rate of $O_p\left(h^{R+1} + \sqrt{\log n/(nh)}\right)$ on the entire support $[0, \overline{x}]$. Note that condition (ii) of $\gamma(\cdot)$ requires the knowledge of $f^{(j)}$, $j = 1, 2, \ldots, R$, which can be challenging in estimation.

6.4 Estimation with transacted bids

We consider an estimation procedure closely following the identification strategy proposed in Section 3.2.

We first recover the marginal bid distributions $G_B(\cdot)$ and $G_S(\cdot)$ on the transacted bids interval $[\underline{s}, \overline{b}]$ as

$$\hat{G}_B(b) = 1 - \widehat{\Pr}(B > b | S = \underline{s}), \quad \hat{G}_S(s) = \widehat{\Pr}(S \leqslant s | B = \overline{b}),$$

where $\widehat{\Pr}(B > b|S = \underline{s})$ and $\widehat{\Pr}(S \leqslant s|B = \overline{b})$ are some smoothing nonparametric estimators. The densities are then estimated by the derivatives as $\hat{g}_B(b) = \hat{G}'_B(b)$ and $\hat{g}_S(s) = \hat{G}'_S(s)$.

In the second step, we recover the corresponding private values $\hat{V}_i$'s (resp. $\hat{C}_i$'s) for the buyer (resp. the seller) by the inverse bidding strategy of (3.5) (resp. (3.6)) for the bids on $[\underline{s}, \overline{b}]$. We can then estimate the conditional value densities $f_{V|V \geqslant \underline{s}}(\cdot)$ and $f_{C|C \leqslant \overline{b}}(\cdot)$ given successful transaction. We can also estimate the conditional value distribution functions $\Pr\left(V \leqslant v \,|\, V \geqslant \underline{s}\right)$, and $\Pr\left(C \leqslant c \,|\, C \leqslant \overline{b}\right)$ according to (3.9). Their asymptotic properties are left for future research.

7 Conclusion

This paper studies nonparametric identification and estimation of double auction with bargaining. It first gives all the restrictions of theoretical model on observed bid distributions, as well as the sharp identified set of unobserved private value distributions when only transacted bids are used. The latter identified set collapses to singleton when the non-transacted bids are also used. We then propose a (boundary and interior) bias corrected two-step estimators of the buyer's and the seller's value densities. The estimators are shown to achieve the optimal convergence rate. Our Monte Carlo experiments demonstrate the significance of the bias correction (especially bias correction in the interior of the support) in the two-step estimation of value densities.

We focus on the identification and estimation of double auction with bargaining. It is interesting to design some nonparametric testing procedures in the context of double auctions, similar to those testing procedures proposed in one-sided auctions, see, e.g., Fang and Tang (2014), Liu and Luo (2017), Liu and Vuong (2021), and Jun and Zincenko (2022).

Appendix

Appendix A collects the proofs of theorems, corollaries, and lemmas in the text. Appendix B presents a supplementary material. Appendix C collects the proofs of results in Appendix B.

A Proofs of Theorems, Corollaries, and Lemmas in the Text

A.1 Proof of Lemma 1

First, we prove that $v > \underline{s}$ implies $\beta_B(v) \leqslant v$.

When $k = 0$, that is, the transaction price is completely determined by the seller's bid, a buyer with private value $v \geqslant \underline{s}$ will get

$$\pi_B(b, v) = \int_{\underline{s}}^{b} (v - s)\, \mathrm{d}G_S(s)$$

from bidding b. Note that the integrand, $v - s$, is strictly decreasing in s, thus

$$\int_{\underline{s}}^{b} (v - s)\, \mathrm{d}G_S(s) \leqslant \int_{\underline{s}}^{+\infty} \max\{v - s, 0\}\, \mathrm{d}G_S(s). \tag{A.1}$$

Since $v > \underline{s}$, the equality in (A.1) holds if $b = v$, and the equality holds for all G_S only if $b = v$. This implies that, when $k = 0$, the truthful strategy $\beta_B(v) = v$ is the unique (weakly) dominant strategy for the buyer.

When $k \in (0, 1]$, we shall show that it is better for the buyer with value $v > \underline{s}$ to bid her value v than any bid $b > v$. Since $\underline{s}$ is the lower bound of the support of G_S, $G_S(\underline{s}) = 0$ and $G_S(v) > 0$, then

$$\begin{aligned}
\pi_B(v, v) - \pi_B(b, v) &= \int_{\underline{s}}^{v} [v - kv - (1-k)s]\, \mathrm{d}G_S(s) - \int_{\underline{s}}^{b} [v - kb - (1-k)s]\, \mathrm{d}G_S(s) \\
&= \int_{\underline{s}}^{v} [v - kv - (1-k)s]\, \mathrm{d}G_S(s) - \int_{\underline{s}}^{v} [v - kb - (1-k)s]\, \mathrm{d}G_S(s) \\
&\qquad - \int_{v}^{b} [v - kb - (1-k)s]\, \mathrm{d}G_S(s) \\
&= \int_{\underline{s}}^{v} k(b - v)\, \mathrm{d}G_S(s) - \int_{v}^{b} [v - kb - (1-k)s]\, \mathrm{d}G_S(s) \\
&= k(b - v)G_S(v) + \int_{v}^{b} [kb + (1-k)s - v]\, \mathrm{d}G_S(s).
\end{aligned}$$

Since $b > v$ and $G_S(v) > 0$, the first term is positive and the second term

$$\int_{v}^{b} [kb + (1-k)s - v]\, \mathrm{d}G_S(s) \geqslant \int_{v}^{b} [kb + (1-k)v - v]\, \mathrm{d}G_S(s) = k(b - v)[G_S(b) - G_S(v)] \geqslant 0.$$

This completes the proof of $\beta_B(v) \leqslant v$.

To see that $\beta_B(v) < v$ for $v > \underline{s}$ if $k > 0$, note that by (2.1),

$$\left.\frac{\partial \pi_B(b, v)}{\partial b}\right|_{b=v} = -kG_S(v) < 0.$$

It implies that there exists $\Delta > 0$ small enough such that $\pi_B(v - \Delta, v) > \pi_B(v, v)$, therefore, bidding the true value for the buyer with private value v is no longer optimal, i.e. $\beta_B(v) \neq v$. Since we have already shown that $\beta_B(v) \leqslant v$, the desired result follows.

In an analogous way, the second conclusion can be proved by showing that truthful bidding strategy is dominant when $k = 1$, and is dominated by some $\tilde{\beta}_S(c) > c$ when $k \in [0, 1)$ and $c < \overline{b}$. $\square$

A.2 Proof of Theorem 1

Let $\beta_B(\cdot)$ and $\beta_S(\cdot)$ be the respective regular equilibrium bidding strategies of the buyer and the seller that induce the bid distribution G.

By condition A1 of Assumption C, strictly increasing and continuous bidding strategies imply the support of bid distribution is a rectangular region, namely $[\underline{b}, \overline{b}] \times [\underline{s}, \overline{s}]$ with $\underline{b} = \beta_B(\underline{v})$, $\overline{b} = \beta_B(\overline{v})$, $\underline{s} = \beta_S(\underline{c})$ and $\overline{s} = \beta_S(\overline{c})$. To show that $\overline{b} \leqslant \overline{s}$ and $\underline{b} \leqslant \underline{s}$, firstly suppose $\overline{b} > \overline{s}$, then any buyer bidding $b > \overline{s}$ will be strictly inferior to just bidding $\overline{s}$. Because this doesn't make the buyer lose any trades but the expected profit on each trade will increase by lowering the transaction price. This deviation is contradicted by the assumption that (β_B, β_S) is an equilibrium. Applying similar argument to the seller bidding $s < \underline{b}$, we can prove the second conclusion $\underline{s} \geqslant \underline{b}$. Then we show that $\underline{s} < \overline{b}$. Suppose not, then: (i) If $\overline{b} \leqslant \underline{s} < \overline{v}$, the buyer with value $\overline{v}$ will have incentive to bid $\frac{\underline{s}+\overline{v}}{2}$ instead of $\overline{b}$, because by bidding $\frac{\underline{s}+\overline{v}}{2}$ he can get

$$\pi\left(\frac{\underline{s}+\overline{v}}{2}, \overline{v}\right) = \int_{\underline{s}}^{\frac{\underline{s}+\overline{v}}{2}} \left[\overline{v} - k\frac{\underline{s}+\overline{v}}{2} - (1-k)s\right] \mathrm{d}G_S(s) = \frac{k}{2}(\overline{v} - \underline{s}) + (1-k)\int_{\underline{s}}^{\frac{\underline{s}+\overline{v}}{2}} (\overline{v} - s)\, \mathrm{d}G_S(s) > 0$$

while bidding $\overline{b} \leqslant \underline{s}$ gives him zero expected profit. This contradicts the equilibrium requirement. (ii) If $\underline{c} < \overline{b} \leqslant \underline{s}$, then analogous argument can show that bidding $\frac{\overline{b}+\underline{c}}{2}$ is a profitable deviation for the seller with value $\underline{c}$, which presents a contradiction to the equilibrium condition, too. (iii) If $\overline{b} \leqslant \underline{c} < \overline{v} \leqslant \underline{s}$, then condition A3 of Assumption C is contradicted because it requires that $\underline{s} = \underline{c} < \overline{v} = \overline{b}$. From the above, C1 hold.

Because V and C are independent and because $\beta_B(\cdot)$ and $\beta_S(\cdot)$ are deterministic functions, it follows that the bids, $B = \beta_B(V)$ and $S = \beta_S(C)$, are also independent. More precisely, since $\beta_B(\cdot)$ and $\beta_S(\cdot)$ are continuous and strictly increasing, so there exist inverse functions, $\beta_B^{-1}(\cdot)$ and $\beta_S^{-1}(\cdot)$, which are also continuous and strictly increasing. Thus

$$\begin{aligned} G(b, s) &= \Pr(\beta_B(V) \leqslant b, \beta_S(C) \leqslant s) \\ &= \Pr(V \leqslant \beta_B^{-1}(b), C \leqslant \beta_S^{-1}(s)) \\ &= \Pr(V \leqslant \beta_B^{-1}(b)) \Pr(C \leqslant \beta_S^{-1}(s)) = F_V(\beta_B^{-1}(b)) F_C(\beta_S^{-1}(s)). \end{aligned}$$

Define

$$G_B(b) = F_V(\beta_B^{-1}(b)) \tag{A.2}$$

$$G_S(s) = F_C(\beta_S^{-1}(s)) \tag{A.3}$$

for every $b \in [\underline{b}, \overline{b}]$ and $s \in [\underline{s}, \overline{s}]$. Since $\beta_B^{-1}(\cdot)$ is continuous and strictly increasing on $[\underline{b}, \overline{b}] =$

$[\beta_B(\underline{v}), \beta_B(\overline{v})]$, we have $G_B \in \mathscr{P}_{[\underline{b},\overline{b}]}$ by (A.2) and the assumption $F_V \in \mathscr{P}_{[\underline{v},\overline{v}]}$. Similar argument can be applied to show $G_S \in \mathscr{P}_{[\underline{s},\overline{s}]}$. Now we get C2.

In order to show C3 and C4, note that $G_B(\cdot)$ and $G_S(\cdot)$ defined in (A.2) and (A.3) must be the distributions of observed (equilibrium) bids of the buyer and the seller, respectively. Now, $\beta_B(\cdot)$ and $\beta_S(\cdot)$ must solve the set of first-order differential equations (3.1) and (3.2). Since (3.3) and (3.4) follow from (3.1) and (3.2), then $\beta_B(\cdot)$ and $\beta_S(\cdot)$ must satisfy

$$\xi(\beta_B(v), G_S) = v, \quad \eta(\beta_S(c), G_B) = c$$

for all $v \geqslant \underline{s}$ and all $c \leqslant \overline{b}$. Noting that $\underline{s} = \beta_S(\underline{c})$ and $\overline{b} = \beta_B(\overline{v})$ and making the change of variable $v = \beta_B^{-1}(b)$ and $c = \beta_S^{-1}(s)$, we obtain

$$\xi(b, G_S) = \beta_B^{-1}(b) \tag{A.4}$$
$$\eta(s, G_B) = \beta_S^{-1}(s) \tag{A.5}$$

for all $b, s \in [\underline{s}, \overline{b}]$. By condition A1 of Assumption C, both $\beta_B^{-1}(\cdot)$ and $\beta_S^{-1}(\cdot)$ are strictly increasing, and by condition A3 ofAssumption C, $\beta_B(\cdot)$ is differentiable on $[\underline{s}, \overline{v}]$ and so is $\beta_S(\cdot)$ on $[\underline{c}, \overline{b}]$. Thus C3 and C4 follow from the fact that $\xi(\underline{s}, G_S) = \underline{s}$ by (3.3), $\eta(\overline{b}, G_B) = \overline{b}$ by (3.4), and $\overline{v} = \beta_B^{-1}(\overline{b}) = \xi(\overline{b}, G_S)$, $\underline{c} = \beta_S^{-1}(\underline{s}) = \eta(\underline{s}, G_B)$.

It is remained to show C5 and C6. Given $b \leqslant \overline{b}$, for buyer with private value v such that $\beta_B(v) = b$, bidding any $b' \in [\overline{b}, \overline{s}]$ should not give him greater profit than bidding b because β_B is the equilibrium bidding strategy for the buyer. That is,

$$\begin{aligned}
0 \geqslant \pi_B(b', v) - \pi_B(b, v) &= \int_{\underline{s}}^{b'} [v - kb' - (1-k)s]\, \mathrm{d}G_S(s) - \int_{\underline{s}}^{b} [v - kb - (1-k)s]\, \mathrm{d}G_S(s) \\
&= v[G_S(b') - G_S(b)] - kb'G_S(b') + kbG_S(b) - (1-k)\int_b^{b'} s\, \mathrm{d}G_S(s) \\
&= k(v - b')G_S(b') - k(v-b)G_S(b) \\
&\quad + (1-k)\left[(v-b')G_S(b') - (v-b)G_S(b) + \int_b^{b'} G_S(s)\, \mathrm{d}s\right] \\
&= (v-b')G_S(b') - (v-b)G_S(b) + (1-k)\int_b^{b'} G_S(s)\, \mathrm{d}s.
\end{aligned}$$

Because $v = \beta_B^{-1}(b) = \xi(b, G_S)$ by (A.4), replacing v by $\xi(b, G_S)$ in the above inequality will yield (3.7). Similarly, for seller with private value c such that $\beta_S(c) = s \geqslant \underline{s}$, using the argument that any deviation of bidding $s' \in [\underline{b}, \underline{s}]$ would not be profitable, we can show that (3.8) must hold. This completes the proof of C6 and the theorem. □

A.3 Proof of Theorem 2

We show the identification of F_V and F_C in two steps. In the first step, we construct a pair of F_V and F_C to rationalize the given G. In the second step, we show that such a pair is unique.

Step 1. To show the sufficiency of C1–C4, define

$$F_V(v) = \begin{cases} G_B(v) & \text{if } v < \underline{s} \\ G_B(\xi^{-1}(v, G_S)) & \text{if } \underline{s} \leqslant v \leqslant \xi(\overline{b}, G_S) \\ 1 & \text{if } v > \xi(\overline{b}, G_S) \end{cases} \tag{A.6}$$

$$F_C(c) = \begin{cases} 0 & \text{if } c < \eta(\underline{s}, G_B) \\ G_S(\eta^{-1}(c, G_B)) & \text{if } \eta(\underline{s}, G_B) \leqslant c \leqslant \overline{b} \\ G_S(c) & \text{if } c > \overline{b} \end{cases} \tag{A.7}$$

and

$$\underline{v} = \underline{b}, \quad \overline{v} = \xi(\overline{b}, G_S), \quad \underline{c} = \eta(\underline{s}, G_B), \quad \overline{c} = \overline{s}.$$

Condition C1 guarantees the functions $\xi(\cdot, G_S)$ in (3.3) and $\eta(\cdot, G_S)$ in (3.4) are well-defined. Since $\underline{b}$ is the lower endpoint of the support of G_B, so for all $v \leqslant \underline{v} = \underline{b}$, $F_V(v) = 0$, and by definition, $F_V(v) = 1$ for all $v > \overline{v} = \xi(\overline{b}, G_S)$. Moreover, because $F_V(\overline{v}) = G_B(\xi^{-1}(\xi(\overline{b}, G_S), G_S)) = G_B(\overline{b}) = 1$, $F_V(\underline{s}) = G_B(\xi^{-1}(\xi(\underline{s}, G_S), G_S)) = G_B(\underline{s})$, G_B is continuous and strictly increasing on $[\underline{b}, \overline{b}]$ by C2, and $\xi^{-1}(\cdot, G_S)$ is continuous and strictly increasing on $[\xi(\underline{s}, G_S), \xi(\overline{b}, G_S)]$ by C3. Then $F_V(\cdot)$ defined by (A.6) is continuous and strictly increasing on $[\underline{b}, \xi(\overline{b}, G_S)] = [\underline{v}, \overline{v}]$. Therefore F_V is a valid absolutely continuous distribution with support $[\underline{v}, \overline{v}]$, i.e. $F_V \in \mathscr{P}_{[\underline{v}, \overline{v}]}$ as required. We can also show $F_C \in \mathscr{P}_{[\underline{c}, \overline{c}]}$ in similar way.

We shall show that the distributions F_V and F_C of buyer's and seller's respective private values can rationalize G in a sealed-bid k-double auction, i.e. $G_B(b) = F_V(\beta_B^{-1}(b))$ on $[\underline{b}, \overline{b}]$ and $G_S(s) = F_C(\beta_S^{-1}(s))$ on $[\underline{s}, \overline{s}]$ for some regular equilibrium profile (β_B, β_S). By construction of F_V and F_C, we have

$$\begin{aligned} G_B(b) &= F_V(b)\mathbb{1}(\underline{b} \leqslant b < \underline{s}) + F_V(\xi(b, G_S))\mathbb{1}(\underline{s} \leqslant b \leqslant \overline{b}) \\ &= F_V\left(b\mathbb{1}(\underline{b} \leqslant b < \underline{s}) + \xi(b, G_S)\mathbb{1}(\underline{s} \leqslant b \leqslant \overline{b})\right) \end{aligned}$$

for $b \in [\underline{b}, \overline{b}]$ and

$$\begin{aligned} G_S(s) &= F_C(\eta(s, G_B))\mathbb{1}(\underline{s} \leqslant s \leqslant \overline{b}) + F_C(s)\mathbb{1}(\overline{b} < s \leqslant \overline{s}) \\ &= F_C\left(\eta(s, G_B)\mathbb{1}(\underline{s} \leqslant s \leqslant \overline{b}) + s\mathbb{1}(\overline{b} < s \leqslant \overline{s})\right) \end{aligned}$$

for $s \in [\underline{s}, \overline{s}]$, where $\mathbb{1}(\cdot)$ is the indicator function. Define

$$\begin{aligned} \xi_*(b, G_S) &\equiv b\mathbb{1}(\underline{b} \leqslant b < \underline{s}) + \xi(b, G_S)\mathbb{1}(\underline{s} \leqslant b \leqslant \overline{b}), \\ \eta_*(s, G_B) &\equiv \eta(s, G_B)\mathbb{1}(\underline{s} \leqslant s \leqslant \overline{b}) + s\mathbb{1}(\overline{b} < s \leqslant \overline{s}), \end{aligned}$$

then by C3 and C4, $\xi_*(\cdot, G_S)$ is continuous and strictly increasing on $[\underline{b}, \overline{b}]$ and so is $\eta_*(\cdot, G_B)$ on $[\underline{s}, \overline{s}]$.

Define bidding strategies

$$\beta_B(v) = \begin{cases} v & \text{if } \underline{v} \leqslant v \leqslant \underline{s} \\ \xi^{-1}(v, G_S) & \text{if } \underline{s} < v \leqslant \overline{v} \end{cases} \tag{A.8}$$

$$\beta_S(c) = \begin{cases} \eta^{-1}(c, G_B) & \text{if } \underline{c} \leqslant c < \overline{b} \\ c & \text{if } \overline{b} \leqslant c \leqslant \overline{c} \end{cases} \tag{A.9}$$

so that $\beta_B(\cdot) = \xi_*^{-1}(\cdot, G_S)$ and $\beta_S(\cdot) = \eta_*^{-1}(\cdot, G_B)$. By construction of these strategies, A1–A3 in Assumption C are satisfied, and also, $G_B(b) = F_V(\beta_B^{-1}(b))$ and $G_S(s) = F_C(\beta_S^{-1}(s))$ so that G is the induced bid distribution for (F_V, F_C) defined in (A.6) and (A.7) by the strategy profile (β_B, β_S) defined above. Thus it remains to show (β_B, β_S) is indeed an equilibrium. We show that the optimal bid for the buyer with private value v is $\beta_B(v)$. A similar argument shows that β_S is optimal for the seller.

Obviously, if $v \leqslant \underline{s}$, then the buyer cannot make an advantageous trade and bidding $\beta_B(v) = v$ achieves zero as her greatest possible expected profit. Suppose $v > \underline{s}$, since G_S is the induced seller's bid distribution, then for bid $b \in [\underline{s}, \overline{b}]$, by (2.1) we obtain

$$\begin{aligned} \frac{\partial \pi_B(b,s)}{\partial b} &= -kG_S(b) + (v - kb)g_S(b) - (1-k)bg_S(b) \\ &= g_S(b)\left[v - \left(b + k\frac{G_S(b)}{g_S(b)}\right)\right] = g_S(b)\left[v - \xi(b, G_S)\right]. \end{aligned}$$

Because $g_S(b)$ is positive, the monotonicity of $\xi(\cdot, G_S)$ by C3 implies that $\partial \pi_B(b,v)/\partial b > 0$ for all $b < \xi^{-1}(v, G_S)$ and $\partial \pi_B(b,v)/\partial b < 0$ for all $b > \xi^{-1}(v, G_S)$. Therefore, $b = \xi^{-1}(v, G_S) = \beta_B(v)$ is the unique maximizer of the buyer's expected profit in $[\underline{s}, \overline{b}]$. Now we show that the buyer would not want to choose bid within $[\overline{b}, \overline{s}]$, either. Recall that we have already shown that C5 is equivalent to $\pi_B(b', v) \leqslant \pi_B(b, v)$ for any $v \geqslant \overline{b}$ and any $b' \in [\overline{b}, \overline{s}]$ when $b = \xi^{-1}(v, G_S) = \beta_B(v)$ in the proof of Theorem 1, this is established straightforwardly because choosing a bid within $[\overline{b}, \overline{s}]$ is profitable only for the buyer with private value $v \geqslant \overline{b}$. Finally, given $\overline{s}$ is the highest seller's bid, any buyer's bid greater than $\overline{s}$ will be dominated by $\overline{s}$. Hence, F_V and F_C indeed rationalize G in a sealed-bid k-double auction.

Step 2. From the proof of Theorem 1, we know that $\xi(\cdot, G_S) = \beta_B^{-1}(\cdot)$ and $\eta(\cdot, G_B) = \beta_S^{-1}(\cdot)$ on $[\underline{s}, \overline{b}]$ when $F_V(\cdot)$ and $F_C(\cdot)$ exist. Since $F_V(\cdot) = G_B(\beta_B(\cdot))$ and $F_C(\cdot) = G_S(\beta_S(\cdot))$, then $F_V(\cdot) = G_B(\xi_*^{-1}(\cdot, G_S))$ and $F_C(\cdot) = G_S(\eta_*^{-1}(\cdot, G_B))$. Because $\xi(\cdot, G_S)$ is uniquely determined by $G_S(\cdot)$ and $\eta(\cdot, G_B)$ is uniquely determined by $G_B(\cdot)$, it follows that $\xi_*(\cdot, G_S)$ and $\eta_*(\cdot, G_B)$ are uniquely determined by G. Hence, the private value distribution (F_V, F_C) that rationalizes G is unique.

This hence establishes the identification of F_V and F_C from any given $G \in \mathscr{P}_{\mathscr{D}}$ satisfying C1–C6. □

A.4 Proof of Lemma 2

It is straightforward to see that conditions C3–C4 are implied by C7–C8. It then suffices to show that C5 and C6 are implied by C7 and C8.

We shall only show C7, more precisely, the monotonicity of $\xi(\cdot, G_S)$, implies C5. A similar argument can show that C8 implies C6. For buyer with private value v, since

$$\frac{\partial \pi_B(b,v)}{\partial b} = g_S(b)\left[v - \xi(b, G_S)\right],$$

then strictly increasing $\xi(\cdot, G_S)$ on $[\underline{s}, \overline{s}]$ ensures that for any $b \in \left(\xi^{-1}(v, G_S), \overline{s}\right]$, $\partial \pi_B(b,v)/\partial b < 0$, therefore, the expected profit of the buyer $\pi_B(b,v)$ is strictly decreasing in the buyer's bid. For $b' \in [\overline{b}, \overline{s}]$ and $b \leqslant \overline{b}$ such that $\xi(b, G_S) \geqslant b'$, let $v = \xi(b, G_S)$, then it follows from the above conclusion that

$$b' \geqslant \overline{b} \geqslant b = \xi^{-1}(v, G_S) \quad \Rightarrow \quad \pi_B(b', v) \leqslant \pi_B(b, v),$$

which is equivalent to C5 as shown in the proof of Theorem 1. □

A.5 Proof of Corollary 1

By Theorem 1, C1–C4 hold. Let $m' = \Pr(\underline{s} \leqslant S \leqslant B \leqslant \overline{b})$. By definition of G_2, D1 is the direct corollary of C1. Using $g_2(b,s) = g(b,s)/m'$ and $g(b,s) = g_B(b)g_S(s)$ by C2, we have

$$g_2(b,s)g_2(b',s') = g_2(b,s')g_2(b',s) = \frac{g_B(b)g_B(b')g_S(s)g_S(s')}{m'^2},$$

so D2 holds. D3 and D4 are implied by C3 and C4, respectively. □

A.6 Proof of Corollary 2

By condition D1 and Lemma 1, we have $\underline{c} \leqslant \underline{s} < \overline{b} \leqslant \overline{v}$, namely condition E1 holds.

Notice that, by D3 and D4, (3.9) is equivalent to

$$\frac{F_V(\xi(b,G_S)) - F_V(\underline{s})}{1 - F_V(\underline{s})} = \frac{G_B(b) - G_B(\underline{s})}{1 - G_B(\underline{s})}, \quad \frac{F_C(\eta(s,G_B))}{F_C(\overline{b})} = \frac{G_S(s)}{G_S(\overline{b})} \tag{A.10}$$

for $(b,s) \in [\underline{s}, \overline{b}]^2$.

We next establish condition E2 by showing (A.10). According to the proof of Theorem 2, G can only be rationalized by (F_V, F_C) defined in (A.6) and (A.7) which imply

$$F_V(\xi(b,G_S)) = G_B(b), \quad F_C(\eta(s,G_B)) = G_S(s) \tag{A.11}$$

for $\underline{s} = \xi^{-1}(\underline{s}, G_S) \leqslant b \leqslant \overline{b}$ and $\underline{s} \leqslant s \leqslant \eta^{-1}(\overline{b}, G_B) = \overline{b}$. By (A.11) and using $\xi(\underline{s}, G_S) = \underline{s}$, $\eta(\overline{b}, G_B) = \overline{b}$, we have condition (A.10) to hold for all F_V and F_C. We hence establish condition E2.

In addition, according to Theorem 1, G satisfies conditions C5 and C6. Given the equilibrium strategies are regular, we have $G_S(s) = F_C(s)$ for all $s > \overline{b}$ and $G_S(s) = F_C(\eta(s, G_B))$ for all $s \leqslant \overline{b}$, therefore, (3.10) immediately follows from (3.7). A similar argument can show (3.11) follows from (3.8), too. We therefore establish condition E3.

The sharpness of identified set is implied by the rationalization result of Corollary 1. This completes the whole proof. □

A.7 Proof of Lemma 3

First, we will establish the following two properties on bidding strategies: (M1) under Assumption F, any regular equilibrium strategies β_B and β_S are twice continuously differentiable on $[\underline{s}, \overline{v}]$ and $[\underline{c}, \overline{b}]$, respectively; (M2) for any $v \in [\underline{s}, \overline{v}]$ and any $c \in [\underline{c}, \overline{b}]$, $\beta_B'(v) \geqslant \epsilon_B > 0$ and $\beta_S'(c) \geqslant \epsilon_S > 0$. To show (M1), we need to rewrite (3.1) and (3.2) as follows:

$$\beta_S'(c) = \frac{f_C(c)\left[\beta_B^{-1}(\beta_S(c)) - \beta_S(c)\right]}{k \cdot F_C(c)}, \tag{A.12}$$

$$\beta_B'(v) = \frac{f_V(v)\left[\beta_B(v) - \beta_S^{-1}(\beta_B(v))\right]}{(1-k) \cdot [1 - F_V(v)]}. \tag{A.13}$$

By definition, any pair of regular equilibrium strategies β_B and β_S is continuously differentiable on $[\underline{s}, \overline{v}]$ and $[\underline{c}, \overline{b}]$, respectively (see Assumption C). Consequently, under Assumption F, (A.12) and (A.13) imply that $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ are continuously differentiable on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$, respectively. This further implies that β_S and β_B are twice continuously differentiable on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$. This completes the proof of (M1).

Now we establish (M2). By definition of regular equilibrium, the seller's and buyer's bidding strategies are continuously differentiable with positive derivative on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$, respectively (see condition A2 of Assumption C), i.e., $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ are continuous and positive on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$. By extreme value theorem, $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ have positive minimum and maximum on $[\underline{c}, \overline{b}]$ and $[\underline{s}, \overline{v}]$, respectively. The conclusion of (M2) therefore follows.

By (A.4) and (A.5), conditions (M1) and (M2) imply that both $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$ are twice continuously differentiable on $[\underline{s}, \overline{b}]$. Note that

$$g_B(b) = \frac{f_V(\beta_B^{-1}(b))}{\beta_B'(\beta_B^{-1}(b))}, \quad g_S(s) = \frac{f_C(\beta_S^{-1}(s))}{\beta_S'(\beta_S^{-1}(s))}.$$

In addition, f_V and f_C are bounded away from 0 by Assumption F, and β_B' and β_S' are bounded away from 0 by (M2). The conclusion of part (i) then follows. Because $G_B(b) = F_V(\beta_B^{-1}(b)) = F_V(\xi(b, G_S))$ for $b \in [\underline{s}, \overline{b}]$, the result about G_B in part (ii) follows from that both $F_V(\cdot)$ and $\xi(\cdot, G_S)$ are twice continuously differentiable on $[\underline{s}, \overline{b}]$. The result about G_S in part (ii) can be shown similarly. Lastly, to prove part (iii), we note that (3.3) and (3.4) give

$$g_S(s) = k \frac{G_S(s)}{\xi(s, G_S) - s}, \quad g_B(b) = (1-k) \frac{1 - G_B(b)}{b - \eta(b, G_B)}.$$

Since every term on the right-hand side is twice continuously differentiable, the desired conclusion follows. $\square$

A.8 Proof of Lemma 4

For part (i), since $\hat{\underline{s}} \geqslant \underline{s}$, $\hat{\bar{b}} \leqslant \bar{b}$ and $|\hat{\underline{s}} - \underline{s}| = O_p(1/n)$, $|\hat{\bar{b}} - \bar{b}| = O_p(1/n)$, the estimation error of $\hat{\underline{s}}$ and $\hat{\bar{b}}$ is negligible. Similar to Theorem 2.1 of Karunamuni and Zhang (2008), it can be shown that on $[\underline{s}, \bar{b}]$,

$$\sup_{b\in[\underline{s},\bar{b}]} |\hat{g}_B(b) - g_B(b)| = O_p\big(h_B^2 + \sqrt{\log n/(nh_B)}\big), \quad \sup_{s\in[\underline{s},\bar{b}]} |\hat{g}_S(s) - g_S(s)| = O_p\big(h_S^2 + \sqrt{\log n/(nh_S)}\big),$$

where both convergence rates can be simplified as $O_p\big((\log n/n)^{2/5}\big)$ under Assumption H. Although g_B (or g_S) is discontinuous at $\underline{s}$ (or $\bar{b}$), we can similarly use the boundary-corrected density kernel estimator to estimated g_B (or g_S) on interval $[\underline{b}, \underline{s}]$ (or interval $[\bar{b}, \bar{s}]$) and with the same argument we can get that $\hat{g}_B$ (or $\hat{g}_S$) converges to the true density at the same rate as on interval $[\underline{s}, \bar{b}]$, then the desired uniform consistency results on the whole support of g_B or g_S follow.

We next show part (ii). We shall show the convergence rate of $\sup_i |\hat{V}_i - V_i|$. The result for $\sup_i |\hat{C}_i - C_i|$ can be shown analogously.

It follows from the definition of $\xi(b, G_S)$ and (4.1) that

$$\begin{aligned}
\mathbb{1}(V_i \in [\underline{s}, \bar{v}])|\hat{V}_i - V_i| &= \mathbb{1}(B_i \in [\underline{s}, \bar{b}]) \cdot k\Big|\frac{\hat{G}_S(B_i)}{\hat{g}_S(B_i)} - \frac{G_S(B_i)}{g_S(B_i)}\Big| \\
&= \mathbb{1}(B_i \in [\underline{s}, \bar{b}]) \cdot k\Big|\frac{\hat{G}_S(B_i) - G_S(B_i)}{g_S(B_i)} - \frac{G_S(B_i)}{g_S(B_i)^2}[\hat{g}_S(B_i) - g_S(B_i)] \\
&\qquad + o\big(\hat{G}_S(B_i) - G_S(B_i)\big) + o\big(\hat{g}_S(B_i) - g_S(B_i)\big)\Big| \\
&\leqslant \mathbb{1}(B_i \in [\underline{s}, \bar{b}])\Big\{\frac{|\hat{G}_S(B_i) - G_S(B_i)|}{g_S(B_i)} + \frac{G_S(B_i)}{g_S(B_i)^2}|\hat{g}_S(B_i) - g_S(B_i)| \\
&\qquad + o\big(|\hat{G}_S(B_i) - G_S(B_i)|\big) + o\big(|\hat{g}_S(B_i) - g_S(B_i)|\big)\Big\} \\
&\leqslant \sup_{B_i\in[\underline{s},\bar{b}]} \Big\{\frac{|\hat{G}_S(B_i) - G_S(B_i)|}{g_S(B_i)} + \frac{G_S(B_i)}{g_S(B_i)^2}|\hat{g}_S(B_i) - g_S(B_i)| \\
&\qquad + o\big(|\hat{G}_S(B_i) - G_S(B_i)|\big) + o\big(|\hat{g}_S(B_i) - g_S(B_i)|\big)\Big\} \\
&\leqslant \frac{\sup_{b\in[\underline{s},\bar{b}]} |\hat{G}_S(b) - G_S(b)|}{\alpha_S} + \frac{1}{\alpha_S^2}\sup_{b\in[\underline{s},\bar{b}]} |\hat{g}_S(b) - g_S(b)| \\
&\qquad + o\big(\sup_{b\in[\underline{s},\bar{b}]} |\hat{G}_S(b) - G_S(b)|\big) + o\big(\sup_{b\in[\underline{s},\bar{b}]} |\hat{g}_S(b) - g_S(b)|\big).
\end{aligned} \tag{A.14}$$

where the last inequality holds since, for any b, $g_S(b) \geqslant \alpha_S$ and $G_S(b) \leqslant 1$. Then,

$$\begin{aligned}
\sup_i \mathbb{1}(V_i \in [\underline{s}, \bar{v}])|\hat{V}_i - V_i| \leqslant &\frac{\sup_{b\in[\underline{s},\bar{b}]} |\hat{G}_S(b) - G_S(b)|}{\alpha_S} + \frac{1}{\alpha_S^2}\sup_{b\in[\underline{s},\bar{b}]} |\hat{g}_S(b) - g_S(b)| \\
&+ o\big(\sup_{b\in[\underline{s},\bar{b}]} |\hat{G}_S(b) - G_S(b)|\big) + o\big(\sup_{b\in[\underline{s},\bar{b}]} |\hat{g}_S(b) - g_S(b)|\big).
\end{aligned}$$

Given that $\sup_{b\in[\underline{s},\overline{b}]}|\hat{G}_S(b)-G_S(b)| \leqslant \sup_{b\in\mathbb{R}}|\hat{G}_S(b)-G_S(b)| = O_p(\log n/\sqrt{n})$, it follows from part (i) that

$$\sup_i \mathbb{1}(V_i\in[\underline{s},\overline{v}])|\hat{V}_i - V_i| = O_p\big((\log n/n)^{2/5}\big). \tag{A.15}$$

Since by regular equilibrium assumption, the buyer with private value $v<\underline{s}$ will bid $b=v$ and hence $\hat{V}_i = B_i = V_i$. Then we can extend the result in (A.15) to all $V_i\in[\underline{v},\overline{v}]$ so that

$$\sup_i|\hat{V}_i - V_i| = \sup_i \mathbb{1}(V_i\in[\underline{s},\overline{v}])|\hat{V}_i - V_i| = O_p\big((\log n/n)^{2/5}\big).$$

This completes the whole proof. □

A.9 Proof of Theorem 3

We shall only show the uniform convergence result of $|\hat{f}_V(\cdot)-f_V(\cdot)|$. The result of $|\hat{f}_C(\cdot)-f_C(\cdot)|$ can be shown similarly, and is hence omitted.

Let $\mathscr{C}_V$ be a closed inner subset of $[\underline{v},\overline{v}]$, and $\tilde{f}_V(\cdot)$ be the (infeasible) one-step boundary-corrected kernel density estimator which uses the unobserved true private values V_i instead of $\hat{V}_i$. Applying similar argument to establish part (i) of Lemma 4, we can show that $\sup_{v\in[\underline{v},\overline{v}]}|\tilde{f}_V(v)-f_V(v)| = O_p\left((\log n/n)^{1/5}\right)$ given a bandwidth $h_V=\lambda_V(\log n/n)^{1/5}$. Since $\hat{f}_V(v)-f_V(v) = [\hat{f}_V(v)-\tilde{f}_V(v)] + [\tilde{f}_V(v)-f_V(v)]$, it remains to show that $\sup_{v\in\mathscr{C}_V}\big|\hat{f}_V(v)-\tilde{f}_V(v)\big| = O_p\big((\log n/n)^{1/5}\big)$.

Let $\mathscr{C}'_V = \bigcup_{v\in\mathscr{C}_V}[v-\Delta,v+\Delta]$ and $\mathscr{C}''_V = \bigcup_{v\in\mathscr{C}'_V}[v-\Delta,v+\Delta]$ for some $\Delta>0$. By construction, $\mathscr{C}'_V$ and $\mathscr{C}''_V$ are also closed, and $\mathscr{C}_V\subset\mathscr{C}'_V\subset\mathscr{C}''_V$. Since $\mathscr{C}_V$ is a closed inner subset of $[\underline{v},\overline{v}]$, Δ can be chosen small enough such that $\mathscr{C}''_V\subset[\underline{v},\overline{v}]$. Now by part (ii) of Lemma 4, for $v\in\mathscr{C}_V$ and n large enough, $\hat{f}_V(v)$ uses at most observations $\hat{V}_i$ in $\mathscr{C}'_V$ and for which V_i is in $\mathscr{C}''_V$. Because for any $v\in\mathscr{C}_V$, $\tilde{f}_V(v)$ uses at most V_i in $\mathscr{C}''_V$ and both $\hat{f}_V(v)$ and $\tilde{f}_V(v)$ are numerically identical to the standard kernel density estimator, we obtain

$$\hat{f}_V(v)-\tilde{f}_V(v) = \frac{1}{nh_V}\sum_{i=1}^n \mathbb{1}(V_i\in\mathscr{C}''_V)\big[K_V(\frac{v-\hat{V}_i}{h_V}) - K_V(\frac{v-V_i}{h_V})\big].$$

A second-order Taylor expansion gives

$$\begin{aligned}\big|\hat{f}_V(v)-\tilde{f}_V(v)\big| = \big|&\frac{1}{nh_V}\sum_{i=1}^n\big[\mathbb{1}(V_i\in\mathscr{C}''_V)(\hat{V}_i-V_i)\cdot\frac{1}{h_V}K'_V(\frac{v-V_i}{h_V})\big]\\ &+\frac{1}{2nh_V}\sum_{i=1}^n\big[\mathbb{1}(V_i\in\mathscr{C}''_V)(\hat{V}_i-V_i)^2\cdot\frac{1}{h_V^2}K''_V(\frac{v-\tilde{V}_i}{h_V})\big]\big|\end{aligned}$$

where $\tilde{V}_i$ is some point between $\hat{V}_i$ and V_i. By triangular inequality,

$$\begin{aligned}\big|\hat{f}_V(v)-\tilde{f}_V(v)\big| \leqslant &\frac{1}{nh_V^2}\sum_{i=1}^n\mathbb{1}(V_i\in\mathscr{C}''_V)\big|\hat{V}_i-V_i\big|\cdot\big|K'_V(\frac{v-V_i}{h_V})\big|\\ &+\frac{1}{2nh_V^3}\sum_{i=1}^n\mathbb{1}(V_i\in\mathscr{C}''_V)\big(\hat{V}_i-V_i\big)^2\cdot\big|K''_V(\frac{v-\tilde{V}_i}{h_V})\big|.\end{aligned} \tag{A.16}$$

Because $\left|K_V''\left(\frac{v-\tilde{V}_i}{h_V}\right)\right| \leqslant \sup_u |K_V''(u)|$, then the right-hand side of (A.16) is bounded by

$$\frac{1}{h_V}\sup_i \mathbb{1}(V_i \in \mathscr{C}_V'')\big|\hat{V}_i - V_i\big| \cdot \frac{1}{nh_V}\sum_{i=1}^n \big|K_V'\big(\frac{v-V_i}{h_V}\big)\big| + \frac{1}{2h_V^3}\sup_i \mathbb{1}(V_i \in \mathscr{C}_V'')\big|\hat{V}_i - V_i\big|^2 \cdot \sup_u |K_V''(u)|.$$

By part (ii) of Lemma 4 and Assumption H,

$$\big|\hat{f}_V(v) - \tilde{f}_V(v)\big| \leqslant O_p\big((\log n/n)^{1/5}\big) \cdot \frac{1}{nh_V}\sum_{i=1}^n \big|K_V'\big(\frac{v-V_i}{h_V}\big)\big| + O_p\big((\log n/n)^{1/5}\big) \cdot \sup_u |K_V''(u)|. \tag{A.17}$$

It can be shown that $\frac{1}{nh_V}\sum_{i=1}^n \big|K_V'\big(\frac{v-V_i}{h_V}\big)\big|$ converges uniformly to $f_V(v)\int_{-\infty}^{\infty}|K_V'(u)|\,\mathrm{d}u$ thus it is bounded uniformly. Moreover, $\sup_u |K_V''(u)| < \infty$ by Assumption G. It then follows that $\sup_{v\in\mathscr{C}_V} |\hat{f}_V(v) - \tilde{f}_V(v)| = O_p\big((\log n/n)^{1/5}\big)$. The desired conclusion therefore follows. □

B Supplementary Material

B.1 Identification of Pricing Weight k from Quantiles of Transaction Price

Let $\Psi_k(p) \equiv \Pr(P \leqslant p)$ be the distribution function of transaction price, where the subscript k indicates the value of this function could also depend on the pricing weight k. Since $\Psi_k(p) = \Pr(kB + (1-k)S \leq p \,|\, \underline{s} \leq S \leq B \leq \overline{b})$, for $0 < k < 1$, we have

$$\Psi_k(p) = \begin{cases} \displaystyle\int_{\underline{s}}^{p}\int_{s}^{\frac{p-(1-k)s}{k}} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s, & \text{if } p \leq k\overline{b} + (1-k)\underline{s}, \\ \displaystyle 1 - \int_{p}^{\overline{b}}\int_{\frac{p-kb}{1-k}}^{b} g_2(b,s)\,\mathrm{d}s\,\mathrm{d}b, & \text{if } p > k\overline{b} + (1-k)\underline{s}, \end{cases} \tag{B.1}$$

where the density function $g_2(b,s) = g(b,s)/\Pr(\underline{s} \leqslant S \leqslant B \leqslant \overline{b})$ is the joint density of transacted bids. When $k = 0$, since $P = S$,

$$\Psi_0(p) = \int_{\underline{s}}^{p}\int_{s}^{\overline{b}} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s, \tag{B.2}$$

and similarly, when $k = 1$,

$$\Psi_1(p) = \int_{\underline{s}}^{p}\int_{\underline{s}}^{b} g_2(b,s)\,\mathrm{d}s\,\mathrm{d}b = \int_{\underline{s}}^{p}\int_{s}^{p} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s. \tag{B.3}$$

In order to establish the conditions on recovering k from the distributions of bids and price, we firstly show the following lemma.

Lemma 5. *For any fixed $p \in (\underline{s}, \overline{b})$, $\Psi_k(p)$ is continuous and strictly decreasing in $k \in [0,1]$.*

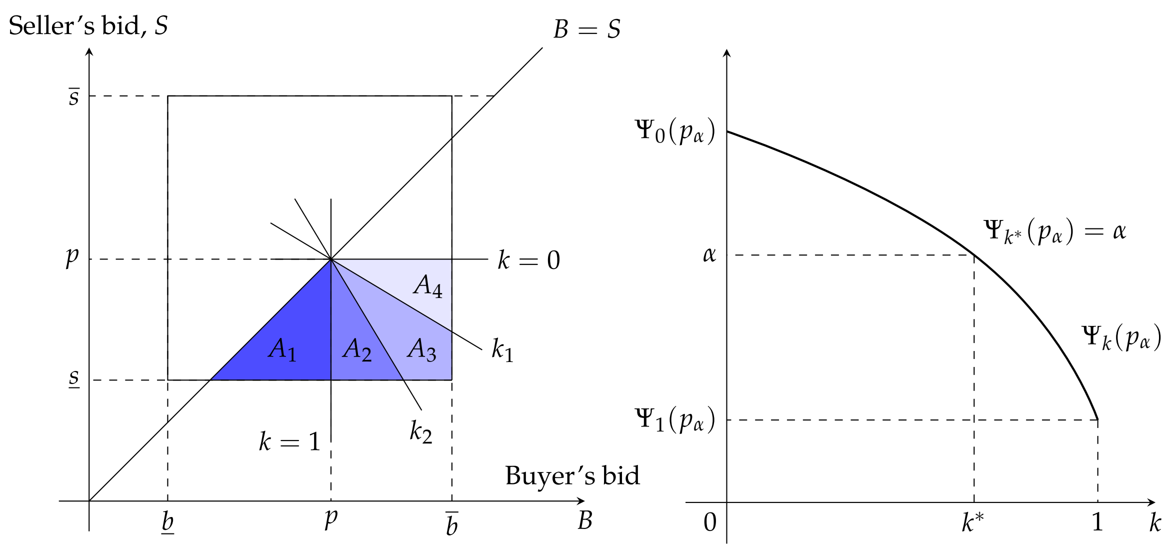
Proof. See Appendix C.1. □

The intuition behind Lemma 5 is given in Figure 4a. This lemma implies that the distribution function (and hence the quantile function) of transaction price is continuous and strictly monotonic in k. If we know some αth-quantile of the transaction price P, say p_α, such that $\underline{s} < p_\alpha < \overline{b}$ and

$\Psi_1(p_\alpha) \leq \alpha \leq \Psi_0(p_\alpha)$, then by Lemma 5, there exists a unique $k^* \in [0,1]$ such that

$$\Psi_{k^*}(p_\alpha) = \alpha. \tag{B.4}$$

Thus, the value of k can be obtained by solving equation (B.4) for k^*.[21] Such an idea is shown by Figure 4b.



(a) Intuition of Lemma 5. Here $0 < k_1 < k_2 < 1$, then $\Psi_{k_1}(p) = \iint_{A_1 \cup A_2 \cup A_3} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s$, $\Psi_{k_2}(p) = \iint_{A_1 \cup A_2} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s$.

(b) Recovering k from a price quantile p_α.

Figure 4: Identification of pricing weight k from quantiles of transaction price

B.2 Density estimator with bias correction

We give the general definition of our bias corrected kernel density estimator in this section. For a random sample $\{X_1, \ldots, X_n\}$ that is drawn from a distribution F with a continuously differentiable density f and support $[\underline{x}, \overline{x}]$, the boundary corrected kernel density estimator of f on interval $[a_1, a_2] \subseteq [\underline{x}, \overline{x}]$ is defined as

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} \mathbb{1}(a_1 \leqslant X_i \leqslant a_2) \Big[K\big(\frac{x - X_i}{h}\big) + K\big(\frac{x - a_1 + \hat{\gamma}_1(X_i - a_1)}{h}\big) + K\big(\frac{a_2 - x + \hat{\gamma}_2(a_2 - X_i)}{h}\big) \Big], \tag{B.5}$$

where

$$\hat{\gamma}_1(u) = u + \hat{d}_1 u^2 + A\hat{d}_1^2 u^3, \quad \hat{\gamma}_2(u) = u + \hat{d}_2 u^2 + A\hat{d}_2^2 u^3,$$

[21] Notice that, for fixed k and p, $\Psi_k(p)$ is identified from the distribution of transacted bids by (B.1).

with

$$\hat{d}_1 = \frac{1}{h'}\Big\{ \log\big[\frac{1}{nh'}\sum_{i=1}^{n} \mathbb{1}(a_1 \leqslant X_i \leqslant a_2) K(\frac{h' - X_i + a_1}{h'}) + \frac{1}{n^2}\big]$$
$$- \log\big[\max\big(\frac{1}{nh'_0}\sum_{i=1}^{n} \mathbb{1}(a_1 \leqslant X_i \leqslant a_2) K_0(\frac{a_1 - X_i}{h'_0}), \frac{1}{n^2}\big)\big]\Big\},$$
$$\hat{d}_2 = \frac{1}{h'}\Big\{ \log\big[\frac{1}{nh'}\sum_{i=1}^{n} \mathbb{1}(a_1 \leqslant X_i \leqslant a_2) K(\frac{h' + X_i - a_2}{h'}) + \frac{1}{n^2}\big]$$
$$- \log\big[\max\big(\frac{1}{nh'_0}\sum_{i=1}^{n} \mathbb{1}(a_1 \leqslant X_i \leqslant a_2) K_0(\frac{X_i - a_2}{h'_0}), \frac{1}{n^2}\big)\big]\Big\},$$

and

$$K_0(u) = (6 + 18u + 12u^2) \cdot \mathbb{1}(-1 \leqslant u \leqslant 0), \quad h'_0 = \Big[\frac{\big(\int_{-1}^{1} u^2 K(u)\,\mathrm{d}u\big)^2 \cdot \int_{-1}^{0} K_0^2(u)\,\mathrm{d}u}{\big(\int_{-1}^{0} u^2 K_0(u)\,\mathrm{d}u\big)^2 \cdot \int_{-1}^{1} K^2(u)\,\mathrm{d}u}\Big]^{1/5} \cdot h'.$$

B.3 Asymptotic results with smoother private value densities

This section provides supplementary asymptotic results when the private value densities have up to R-th order derivatives for $R \geqslant 2$. These results provide the uniform convergence rates of bid density estimate, pseudo value estimate, and value density estimate on any closed inner subset of a given support under the private value densities smoother than the ones in Section 4. They are parallel to the asymptotic properties of Guerre, Perrigne, and Vuong (2000)'s two-step estimator of private value density in first-price auctions. Note that our estimators (with bias correction) actually degenerate to the estimators without bias correction in the interior of a support when the sample size is large enough, thus all asymptotic results of this section can also be viewed as the ones for the estimators without bias correction.

We state the assumptions under which the supplementary asymptotic results are established.

Assumption F′. *F_V and F_C admit up to $R+1$ continuous bounded derivatives on $[\underline{v}, \overline{v}]$ and $[\underline{c}, \overline{c}]$, respectively. In addition, $f_V(v) \geqslant \alpha_V > 0$ for all $v \in [\underline{v}, \overline{v}]$; $f_C(c) \geqslant \alpha_C > 0$ for all $c \in [\underline{c}, \overline{c}]$.*

Assumption G′. *(i) The kernels K_B, K_S, K_V, K_C are symmetric with support $[-1, 1]$ and have twice continuous bounded derivatives. (ii) K_B, K_S, K_V and K_C are of order $R+1$, $R+1$, R, and R.*

Assumption H′. *The bandwidths h_B, h_S, h_V, h_C are of the form:*

$$h_B = \lambda_B\big(\log n/n\big)^{1/(2R+3)}, h_S = \lambda_S\big(\log n/n\big)^{1/(2R+3)}, h_V = \lambda_V\big(\log n/n\big)^{1/(2R+3)}, h_C = \lambda_C\big(\log n/n\big)^{1/(2R+3)},$$

where the λ's are positive constants. The parameters $A_B, A_S, A_V, A_C > 1/3$ and the secondary bandwidths are of the form:

$$h'_B = \tau_B n^{-1/(2R+3)}, \quad h'_S = \tau_S n^{-1/(2R+3)}, \quad h'_V = \tau_V n^{-1/(2R+3)}, \quad h'_C = \tau_C n^{-1/(2R+3)},$$

where the τ's are positive constants.

Assumptions F to H can be viewed as a special case of Assumptions F′ to H′ with $R = 1$, respectively.

The first lemma generalizes Lemma 3 to the case with private densities admitting up to R-th order derivatives.

Lemma 6. *Given Assumption F′, the distributions of regular equilibrium bids G_B and G_S satisfy:*

(i) for any $b \in [\underline{b}, \overline{b}]$ and any $s \in [\underline{s}, \overline{s}]$, $g_B(b) \geqslant \alpha_B > 0$, $g_S(s) \geqslant \alpha_S > 0$;

(ii) G_B and G_S admit up to $R+1$ continuous bounded derivatives on $[\underline{s}, \overline{b}]$;

(iii) g_B and g_S admit up to $R+1$ continuous bounded derivatives on $[\underline{s}, \overline{b}]$.

The proof of Lemma 6 is a straightforward extension of the proof for Lemma 3 in Appendix A.7, and is hence omitted here.

The second lemma studies the uniform convergence rate of bid density estimator and pseudo private values on a close inner subset of transacted interval.

Lemma 7. *Under Assumption E and Assumptions F′ to H′,*

(i). for any (fixed) closed inner subset $\mathscr{C}_g$ of $[\underline{s}, \overline{b}]$,

$$\sup_{b \in \mathscr{C}_g} |\hat{g}_B(b) - g_B(b)| = O_p\big((\log n/n)^{(R+1)/(2R+3)}\big), \ \sup_{s \in \mathscr{C}_g} |\hat{g}_S(s) - g_S(s)| = O_p\big((\log n/n)^{(R+1)/(2R+3)}\big).$$

(ii). For any (fixed) closed inner subsets $\mathscr{C}_V$ of $[\underline{s}, \overline{v}]$ and $\mathscr{C}_C$ of $[\underline{c}, \overline{b}]$,

$$\sup_i \mathbb{1}(V_i \in \mathscr{C}_V)|\hat{V}_i - V_i| = O_p\big((\log n/n)^{(R+1)/(2R+3)}\big), \ \sup_i \mathbb{1}(C_i \in \mathscr{C}_C)|\hat{C}_i - C_i| = O_p\big((\log n/n)^{(R+1)/(2R+3)}\big).$$

Proof. See Appendix C.3. □

Lemma 7 shows that, if the primary bandwidths h_B and h_S are of order $(\log n/n)^{1/(2R+3)}$ according to Assumption H′, both the bid density estimators and pseudo private values achieve a rate of uniform convergence, $(\log n/n)^{(R+1)/(2R+3)}$, on any closed inner subset of transacted interval.

The third result is about the uniform convergence rate of the two-step estimator of value density on a closed inner subset excluding $\underline{s}$ and $\overline{b}$.

Proposition 1. *Under Assumption E and Assumptions F′ to H′, for any (fixed) closed inner subsets $\mathscr{C}_V$ of $[\underline{v}, \overline{v}] \backslash \{\underline{s}\}$ and $\mathscr{C}_C$ of $[\underline{c}, \overline{c}] \backslash \{\overline{b}\}$,*

$$\sup_{v \in \mathscr{C}_V} |\hat{f}_V(v) - f_V(v)| = O_p\big((\log n/n)^{R/(2R+3)}\big), \quad \sup_{c \in \mathscr{C}_C} |\hat{f}_C(c) - f_C(c)| = O_p\big((\log n/n)^{R/(2R+3)}\big).$$

Proof. See Appendix C.4. □

Proposition 1 establishes the uniform consistency of our two-step estimator of the bidders' private value density on any closed inner subset of value support excluding $\underline{s}$ (or $\overline{b}$). The rate of convergence coincides with the optimal convergence rate of Guerre, Perrigne, and Vuong (2000) for the first-price auctions. However, it does not provide the uniform convergence rate of value density on a closed inner subset containing $\underline{s}$ (or $\overline{b}$), although the value density is continuous at this interior point $\underline{s}$ (or $\overline{b}$).

C Proofs of Lemmas and Proposition in Supplementary Material

C.1 Proof of Lemma 5

First, note that when $k \in (0, 1]$, we can rewrite (B.1) and (B.3) together as

$$\Psi_k(p) = \int_{\underline{s}}^{p} \int_{s}^{\min\left(\frac{p-(1-k)s}{k}, \overline{b}\right)} g_2(b, s)\, \mathrm{d}b\, \mathrm{d}s. \tag{C.1}$$

Keep $p \in (\underline{s}, \overline{b})$ fixed and define a function φ as the inner integral in (C.1), i.e.

$$\varphi(k, s) = \int_{s}^{\min\left(\frac{p-(1-k)s}{k}, \overline{b}\right)} g_2(b, s)\, \mathrm{d}b, \quad k \in (0, 1],\ s \in [\underline{s}, p]. \tag{C.2}$$

Since $g_2(b, s)$ is integrable, so φ is continuous in the upper limit of integral. And since the upper limit, $\min\left(\frac{p-(1-k)s}{k}, \overline{b}\right)$, is continuous in k, so φ is continuous in k. Note that $g_2(b, s) > 0$ because the interval of integration is in the support of G, and note that $\min\left(\frac{p-(1-k)s}{k}, \overline{b}\right) \leq \overline{b}$, thus for any $k \in (0, 1]$,

$$0 \leq \varphi(k, s) \leq \int_{s}^{\overline{b}} g_2(b, s)\, \mathrm{d}b \equiv \bar{\varphi}(s), \quad \forall s \in [\underline{s}, p].$$

Therefore, for any $k \in (0, 1]$, for any sequence $\{k_n\}$ in $(0, 1]$ such that $k_n \to k$ as $n \to \infty$, by continuity of φ in k, we have $\tilde{\varphi}_n(s) \equiv \varphi(k_n, s)$ converges pointwise to $\tilde{\varphi}(s) \equiv \varphi(k, s)$ on $[\underline{s}, p]$. Since $\bar{\varphi}(s)$ is integrable, by dominated convergence theorem, as $n \to \infty$,

$$\int_{\underline{s}}^{p} \tilde{\varphi}_n(s)\, \mathrm{d}s \to \int_{\underline{s}}^{p} \tilde{\varphi}(s)\, \mathrm{d}s,$$

hence, $\Psi_{k_n}(p) \to \Psi_k(p)$.

To see the (right) continuity at $k = 0$, we just need to rewrite (B.1) and (B.2) as

$$\Psi_k(p) = 1 - \int_{p}^{\overline{b}} \int_{\frac{p-kb}{1-k}}^{b} g_2(b, s)\, \mathrm{d}s\, \mathrm{d}b, \quad 0 \leq k < \frac{p - \underline{s}}{\overline{b} - \underline{s}}$$

and define

$$\psi(k, b) = -\int_{\frac{p-kb}{1-k}}^{b} g(b, s)\, \mathrm{d}s, \quad k \in \left[0, \frac{p - \underline{s}}{\overline{b} - \underline{s}}\right),\ b \in [p, \overline{b}].$$

Then applying analogous argument, we have ψ is continuous in k so that for sequence $\{k_n\}$ in $\left[0, \frac{p-\underline{s}}{\overline{b}-\underline{s}}\right)$ such that $k_n \to 0$, the sequence $\{\tilde{\psi}_n(b) \equiv \psi(k_n, b)\}$ converges pointwise to $\tilde{\psi}(b) \equiv \psi(0, b)$. Since $\{\tilde{\psi}_n(b)\}$ is dominated by $\bar{\psi}(b) \equiv \int_{\underline{s}}^{b} g(b, s)\, \mathrm{d}s$, we finally can get $\Psi_{k_n}(p) \to \Psi_0(p)$.

It remains to show the monotonicity of $\Psi_k(p)$ in k. Suppose $0 \leq k_1 < k_2 \leq 1$, then by (B.1), (B.2), and (B.3):

(i) If $k_2 < \frac{p-\underline{s}}{\overline{b}-\underline{s}}$, then

$$\Psi_{k_1}(p) - \Psi_{k_2}(p) = \int_{p}^{\overline{b}} \int_{b-\frac{b-p}{1-k_2}}^{b-\frac{b-p}{1-k_1}} g_2(b, s)\, \mathrm{d}s\, \mathrm{d}b > 0$$

due to $\frac{b-p}{1-k_2} > \frac{b-p}{1-k_1}$.

(ii) If $k_1 \geq \frac{p-\underline{s}}{\overline{b}-\underline{s}}$, then

$$\Psi_{k_1}(p) - \Psi_{k_2}(p) = \int_{\underline{s}}^{p} \int_{s+\frac{p-s}{k_2}}^{s+\frac{p-s}{k_1}} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s > 0$$

due to $\frac{p-s}{k_2} < \frac{p-s}{k_1}$.

(iii) If $k_1 < \frac{p-\underline{s}}{\overline{b}-\underline{s}} \leq k_2$, then

$$\Psi_{k_1}(p) - \Psi_{k_2}(p) = \int_{\underline{s}}^{p} \int_{s+\frac{p-s}{k_2}}^{s+\frac{(p-s)(\overline{b}-\underline{s})}{p-\underline{s}}} g_2(b,s)\,\mathrm{d}b\,\mathrm{d}s + \int_{p}^{\overline{b}} \int_{b-\frac{(b-p)(\overline{b}-\underline{s})}{\overline{b}-p}}^{b-\frac{b-p}{1-k_1}} g_2(b,s)\,\mathrm{d}s\,\mathrm{d}b > 0,$$

where the first term is non-negative and the second one is positive.

□

C.2 Proof of Lemma 6

First, we will establish the following two properties on bidding strategies: (M1) under Assumption F, any regular equilibrium strategies β_B and β_S admit up to $R+1$ continuous and bounded derivatives on $[\underline{s},\overline{v}]$ and $[\underline{c},\overline{b}]$, respectively; (M2) for any $v \in [\underline{s},\overline{v}]$ and any $c \in [\underline{c},\overline{b}]$, $\beta_B'(v) \geqslant \epsilon_B > 0$ and $\beta_S'(c) \geqslant \epsilon_S > 0$. To show (M1), we need to rewrite (3.1) and (3.2) as follows:

$$\beta_S'(c) = \frac{f_C(c)\left[\beta_B^{-1}(\beta_S(c)) - \beta_S(c)\right]}{k \cdot F_C(c)}, \tag{C.3}$$

$$\beta_B'(v) = \frac{f_V(v)\left[\beta_B(v) - \beta_S^{-1}(\beta_B(v))\right]}{(1-k)\cdot[1-F_V(v)]}. \tag{C.4}$$

By definition, any pair of regular equilibrium strategies β_B and β_S is continuously differentiable on $[\underline{s},\overline{v}]$ and $[\underline{c},\overline{b}]$, respectively (see Assumption C). Consequently, under Assumption F', (C.3) and (C.4) imply that $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ are continuously differentiable on $[\underline{c},\overline{b}]$ and $[\underline{s},\overline{v}]$, respectively. This further implies that β_S and β_B are twice continuously differentiable on $[\underline{c},\overline{b}]$ and $[\underline{s},\overline{v}]$. Again, under Assumption F, (A.12) and (A.13) imply that $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ are twice continuously differentiable, and hence β_S and β_B admit up to third continuous bounded derivatives on $[\underline{c},\overline{b}]$ and $[\underline{s},\overline{v}]$, respectively. This argument can go on until we conclude that β_S and β_B admit up to $R+1$ continuous bounded derivatives, respectively, on $[\underline{c},\overline{b}]$ and $[\underline{s},\overline{v}]$. This completes the proof of (M1).

Now we establish (M2). By definition of regular equilibrium, the seller's and buyer's bidding strategies are continuously differentiable with positive derivative on $[\underline{c},\overline{b}]$ and $[\underline{s},\overline{v}]$, respectively (see condition A2 of Assumption C), i.e., $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ are continuous and positive on $[\underline{c},\overline{b}]$ and $[\underline{s},\overline{v}]$. By extreme value theorem, $\beta_S'(\cdot)$ and $\beta_B'(\cdot)$ have positive minimum and maximum on $[\underline{c},\overline{b}]$ and $[\underline{s},\overline{v}]$, respectively. The conclusion of (M2) therefore follows.

It was shown earlier that $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$ solve

$$\forall\, b,s \in [\underline{s},\overline{b}]: \quad \beta_B(\xi(b,G_S)) = b, \quad \beta_S(\eta(s,G_B)) = s,$$

it follows from (M1), (M2) and Lemma C1 of Guerre, Perrigne, and Vuong (2000) that both $\xi(\cdot, G_S)$ and $\eta(\cdot, G_B)$ admit up to $R+1$ continuous and bounded derivatives on $[\underline{s}, \overline{b}]$. Note that

$$g_B(b) = \frac{f_V(\beta_B^{-1}(b))}{\beta_B'(\beta_B^{-1}(b))}, \quad g_S(s) = \frac{f_C(\beta_S^{-1}(s))}{\beta_S'(\beta_S^{-1}(s))}.$$

In addition, f_V and f_C are bounded away from 0 by Assumption F, and β_B' and β_S' are bounded by (M2). The conclusion of part (i) then follows. Because $G_B(b) = F_V(\beta_B^{-1}(b)) = F_V(\xi(b, G_S))$ for $b \in [\underline{s}, \overline{b}]$, the result about G_B in part (ii) follows from that both $F_V(\cdot)$ and $\xi(\cdot, G_S)$ have $R+1$ continuous and bounded derivatives on $[\underline{s}, \overline{b}]$. The result about G_S in part (ii) can be proven similarly. Lastly, to prove part (iii), we note that (3.3) and (3.4) give

$$g_S(s) = k\frac{G_S(s)}{\xi(s, G_S) - s}, \quad g_B(b) = (1-k)\frac{1 - G_B(b)}{b - \eta(b, G_B)}.$$

Since every term on the right-hand side admits up to $R+1$ continuous and bounded derivatives, the desired result follows. □

C.3 Proof of Lemma 7

For part (i), we shall only show the convergence rate of $\sup_{b\in\mathscr{C}_g} |\hat{g}_B(b) - g_B(b)|$. The rate of $\sup_{s\in\mathscr{C}_g} |\hat{g}_S(s) - g_S(s)|$ can be shown similarly.

Note $\hat{\underline{s}} \geqslant \underline{s}$, $\hat{\overline{b}} \leqslant \overline{b}$ and as $n \to \infty$, $\hat{\underline{s}} \xrightarrow{p} \underline{s}$ and $\hat{\overline{b}} \xrightarrow{p} \overline{b}$. Given $\lim_{n\to\infty} h_B = 0$ by Assumption H′, for sufficiently large n, $\mathscr{C}_g \subset [\hat{\underline{s}} + h_B, \hat{\overline{b}} - h_B]$ and therefore the boundary-corrected kernel density estimator $\hat{g}_B$ will be numerically identical to the standard kernel density estimator $\tilde{g}_B$ (without boundary correction). Thus, using the existing results for the standard kernel density estimator (see Li and Racine (2006), page 31, Theorem 1.4), we have under Assumption E and Assumptions F′ to H′,

$$\sup_{b\in\mathscr{C}_g} |\hat{g}_B(b) - g_B(b)| = O_p\left(h_B^{R+1} + \sqrt{\frac{\log n}{nh_B}}\right) = O_p\left(\left(\frac{\log n}{n}\right)^{\frac{R+1}{2R+3}}\right).$$

We next show part (ii). We only establish the convergence rate of $\sup_i \mathbb{1}(V_i \in \mathscr{C}_V)|\hat{V}_i - V_i|$. The case of $\sup_i \mathbb{1}(C_i \in \mathscr{C}_C)|\hat{C}_i - C_i|$ can be shown in a similar way.

Define $\mathscr{C}_B = \{b \in [\underline{s}, \overline{b}] \,|\, \xi(b, G_S) \in \mathscr{C}_V\}$. Because $\xi(\cdot, G_S)$ is a strictly increasing continuous function and $\mathscr{C}_V$ is a closed inner subset of $[\underline{s}, \overline{v}]$, then $\mathscr{C}_B$ is also a (fixed) closed inner subset of $[\underline{s}, \overline{b}]$. Following an argument similar to (A.14) by replacing $[\underline{s}, \overline{v}]$ and $[\underline{s}, \overline{b}]$ with $\mathscr{C}_V$ and $\mathscr{C}_B$, respectively, we can get

$$\mathbb{1}(V_i \in \mathscr{C}_V)|\hat{V}_i - V_i| \leqslant \frac{\sup_{b\in\mathscr{C}_B} |\hat{G}_S(b) - G_S(b)|}{\alpha_S} + \frac{1}{\alpha_S^2}\sup_{b\in\mathscr{C}_B} |\hat{g}_S(b) - g_S(b)|$$
$$+ o\left(\sup_{b\in\mathscr{C}_B} |\hat{G}_S(b) - G_S(b)|\right) + o\left(\sup_{b\in\mathscr{C}_B} |\hat{g}_S(b) - g_S(b)|\right).$$

which implies that

$$\sup_i \mathbb{1}(V_i \in \mathscr{C}_V)|\hat{V}_i - V_i| \leqslant \frac{\sup_{b\in\mathscr{C}_B} |\hat{G}_S(b) - G_S(b)|}{\alpha_S} + \frac{1}{\alpha_S^2} \sup_{b\in\mathscr{C}_B} |\hat{g}_S(b) - g_S(b)| + o\left(\sup_{b\in\mathscr{C}_B} |\hat{G}_S(b) - G_S(b)|\right) + o\left(\sup_{b\in\mathscr{C}_B} |\hat{g}_S(b) - g_S(b)|\right).$$

Since $\sup_{b\in\mathscr{C}_B} |\hat{G}_S(b) - G_S(b)| \leqslant \sup_{b\in\mathbb{R}} |\hat{G}_S(b) - G_S(b)| = O_p(\log n/\sqrt{n})$, the desired result follows from part (i) and $O_p\left(\max\left(\log n/\sqrt{n}, (\log n/n)^{(R+1)/(2R+3)}\right)\right) = O_p\left((\log n/n)^{(R+1)/(2R+3)}\right)$. □

C.4 Proof of Proposition 1

We will only establish the result of $|\hat{f}_V(\cdot) - f_V(\cdot)|$. The case of $|\hat{f}_C(\cdot) - f_C(\cdot)|$ can be shown similarly.

We consider the first case that $\mathscr{C}_V$ is a closed inner subset of $[\underline{s}, \overline{v}]$. Let $\tilde{f}_V(v)$ define the (infeasible) one-step boundary-corrected kernel density estimator which uses the unobserved true private values V_i instead of $\hat{V}_i$. Similar to part (i) of Lemma 7, we can show that $\sup_{v\in\mathscr{C}_V} |\tilde{f}_V(v) - f_V(v)| = O_p\big((\log n/n)^{R/(2R+3)}\big)$ under a bandwidth of $h_V = \lambda_V(\log n/n)^{1/(2R+3)}$. Since $\hat{f}_V(v) - f_V(v) = [\hat{f}_V(v) - \tilde{f}_V(v)] + [\tilde{f}_V(v) - f_V(v)]$, it remains to show $\sup_{v\in\mathscr{C}_V} |\hat{f}_V(v) - \tilde{f}_V(v)| = O_p\big((\log n/n)^{R/(2R+3)}\big)$.

Let $\mathscr{C}_V' = \bigcup_{v\in\mathscr{C}_V}[v-\Delta, v+\Delta]$ and $\mathscr{C}_V'' = \bigcup_{v\in\mathscr{C}_V'}[v-\Delta, v+\Delta]$ for some $\Delta > 0$. Similar to the proof of Theorem 3 in Appendix A.9, for small enough Δ and large enough n, we can obtain the following inequality analogous to (A.16)

$$\left|\hat{f}_V(v) - \tilde{f}_V(v)\right| \leqslant \frac{1}{nh_V^2}\sum_{i=1}^n \mathbb{1}(V_i \in \mathscr{C}_V'')\left|\hat{V}_i - V_i\right| \cdot \left|K_V'\left(\frac{v-V_i}{h_V}\right)\right| + \frac{1}{2nh_V^3}\sum_{i=1}^n \mathbb{1}(V_i \in \mathscr{C}_V'')\left(\hat{V}_i - V_i\right)^2 \cdot \left|K_V''\left(\frac{v-\tilde{V}_i}{h_V}\right)\right|. \tag{C.5}$$

Because $\left|K_V''\left(\frac{v-\tilde{V}_i}{h_V}\right)\right| \leqslant \sup_u |K_V''(u)|$, then the right-hand side of (C.5) is bounded by

$$\frac{1}{h_V}\sup_i \mathbb{1}(V_i \in \mathscr{C}_V'')\left|\hat{V}_i - V_i\right| \cdot \frac{1}{nh_V}\sum_{i=1}^n \left|K_V'\left(\frac{v-V_i}{h_V}\right)\right| + \frac{1}{2h_V^3}\sup_i \mathbb{1}(V_i \in \mathscr{C}_V'')\left|\hat{V}_i - V_i\right|^2 \cdot \sup_u |K_V''(u)|.$$

By part (ii) of Lemma 7 and Assumption H′,

$$\left|\hat{f}_V(v) - \tilde{f}_V(v)\right| \leqslant O_p\left(\left(\frac{\log n}{n}\right)^{\frac{R}{2R+3}}\right) \cdot \frac{1}{nh_V}\sum_{i=1}^n \left|K_V'\left(\frac{v-V_i}{h_V}\right)\right| + O_p\left(\left(\frac{\log n}{n}\right)^{\frac{2R-1}{2R+3}}\right) \cdot \sup_u |K_V''(u)|. \tag{C.6}$$

It can be shown that $\frac{1}{nh_V}\sum_{i=1}^n \left|K_V'\left(\frac{v-V_i}{h_V}\right)\right|$ converges uniformly to $f_V(v)\int_{-\infty}^{\infty}|K_V'(u)|\,\mathrm{d}u$ thus it is bounded uniformly. Moreover, $\sup_u |K_V''(u)| < \infty$ by Assumption G. Since $R \geqslant 1$ implies $\frac{2R-1}{2R+3} \geqslant \frac{R}{2R+3}$, it follows that $\sup_{v\in\mathscr{C}_V} |\hat{f}_V(v) - \tilde{f}_V(v)| = O_p\left((\log n/n)^{R/(2R+3)}\right)$ and therefore $\sup_{v\in\mathscr{C}_V} |\hat{f}_V(v) - f_V(v)| = O_p\left((\log n/n)^{R/(2R+3)}\right)$.

Now we consider the other case that $\mathscr{C}_V$ is a closed inner subset of $[\underline{v}, \underline{s}]$ when $\underline{s} > \underline{v}$. By regular equilibrium assumption, the buyer with private value $v < \underline{s}$ will bid $b = v$, thus we have $\hat{V}_i = B_i = V_i$.

Thus $\hat{f}_V$ is in fact the one-step boundary-corrected kernel estimator for f_V on $[\underline{v}, \underline{s}]$. Similar to part (i) of Lemma 7, we can show that $\sup_{v \in \mathscr{C}_V} |\hat{f}_V(v) - f_V(v)| = O_p\left((\log n/n)^{R/(2R+3)}\right)$.

Since any given closed inner subset $\mathscr{C}_V \subseteq [\underline{v}, \overline{v}] \backslash \{\underline{s}\}$ is a union of at most two closed inner subsets respectively belonging to the two cases above, the desired conclusion therefore follows. □

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